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Abstract

State-dependent pricing models are now an operational framework for quantitative business cycle analysis. The analysis in Ball and Romer [1991], however, suggests that such models may be rife with multiple equilibria: in their static model price adjustment is always characterized by complementarity, a necessary condition for multiplicity. We study existence and uniqueness of equilibrium in a discrete-time state-dependent pricing model. In steady states of our model, we find only weak complementarity and no evidence of multiplicity. We likewise find no evidence of multiplicity in the presence of monetary shocks. However, nonexistence of symmetric steady-state equilibrium with pure strategies arises in a small region of the parameter space.

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1. Introduction

Dynamic models with monopolistic competition and sticky prices are one of the leading areas of quantitative research aimed at understanding the real effects of monetary policy.[1] Such models have emerged from two strands of the literature: the New Keynesian approach, whose hallmark is imperfect competition, and the real business cycle approach, whose hallmark is dynamic general equilibrium. While most recent work does not model the underlying source of stickiness, fixed costs of price adjustment ("menu costs") are a common explanation. Models in which monopolistically competitive firms face such costs hold considerable promise for improving our understanding of business cycles, as well as for evaluating the effects of monetary policy. The recent contribution by Dotsey, King and Wolman [1999] shows that such models are now operational for business cycle and policy analysis.

In dynamic general equilibrium models with fixed costs of price adjustment, little attention has been focused on the existence and uniqueness of equilibrium. Modelers have typically taken for granted that equilibrium exists and is unique, and have used common-sense algorithms to compute equilibrium. Our purpose in this paper is to systematically study the existence and uniqueness of equilibrium in a benchmark discrete-time dynamic general equilibrium model with fixed costs of price adjustment.

The motivation for this work is two-fold. First, the models under consideration are not perfectly competitive and have nonconvexities (fixed costs), so standard existence and uniqueness results do not hold. Given the increasing popularity of these models, it is important for researchers to have some guidance as to whether existence or multiplicity are issues they need to be concerned with. Second, there is already a presumption from Ball and Romer (1991) that models with fixed costs of price adjustment are rife with multiple equilibria. However, since Ball and Romer studied a static model, it remains an open question whether their results apply to the dynamic models currently being used to study real fluctuations and monetary policy.

Our main findings are (i) that multiplicity of equilibria does not appear to occur in our dynamic general equilibrium model, and (ii) that for small regions of the parameter space, symmetric steady state equilibrium does not exist, although a naive approach to computing

[1] See, for example, Woodford [2003].

steady states would erroneously conclude that equilibrium does exist.

The model we analyze is a version of that in Dotsey, King and Wolman [1999]. This is a discrete-time dynamic monopolistic competition model in which firms face a distribution of menu costs, and there is thus an endogenous degree of price stickiness. The framework differs from the seminal menu cost models of Caplin and Spulber [1987] and Caplin and Leahy [1991], and the more recent work by Danziger [1999], in that there is a nondegenerate distribution of fixed costs. The Dotsey-King-Wolman framework also fits more readily into the literature on equilibrium business cycles; if we set the fixed costs of price adjustment to zero, the model is a standard real business cycle model, albeit one in which there is no capital.

Although our work is motivated in part by Ball and Romer's finding, we do not construct an exact dynamic generalization of their model. Our aim is to study a model that fits into current mainstream work on equilibrium business cycles. Ball-Romer used preferences that are not standard: specifically, they assumed that utility was linear in consumption. We use preferences that are logarithmic in consumption. In addition, they adopted a "yeoman-farmer" model, in which agents produce output using their own labor input instead of firms producing output by hiring labor in an economywide market. This has several implications. First, it effectively closes down a general equilibrium linkage in their model. In a world of yeoman farmers, the sole general equilibrium connection across firms is through aggregate demand, whereas in our setting, there is also a linkage through the real wage. A second and related point is that the behavior of marginal cost is affected. In the yeoman-farmer setting, marginal cost varies across producers, because it reflects the disutility of leisure. In our setting marginal cost is common across firms. Third, the yeoman-farmer approach allows Ball-Romer to model the cost of changing prices as a separable utility cost that has no connection to the rest of the model. In our setting, adjustment of prices requires labor, which means that the cost of adjustment is affected by changes in the real wage.

As stated above, Ball and Romer [1991] argued that monopolistic competition models with menu costs tend to exhibit multiplicity in the degree of equilibrium price rigidity. If a firm anticipates that other firms will have sticky prices, then it may find sticky prices to be privately optimal, but if the firm expects others to have flexible prices, then price adjustment may be privately optimal. The extent of price stickiness – and hence the effect of aggregate demand shocks on prices and output – would then be indeterminate.[2] Ball and Romer further argued that multiple equilibria enable these models to better explain phenomena such as the varying degrees of nominal rigidity observed across countries.[3]

Ball and Romer's experiment was a one time change in the money stock, in a model with only one decision period.[4] They found that in the face of such a change, there is complementarity in price-setting: the higher is the total fraction of firms adjusting their prices, the more likely it is that an individual firm will adjust its price (the best-response function for price adjustment slopes up). Such complementarity is a necessary condition for multiple symmetric equilibria. In a dynamic setting, there are several possible counterparts to the multiple equilibria discussed by Ball and Romer. First, a dynamic model might possess multiple steady states, so that economies identical in terms of their primitives might exhibit different degrees of price rigidity in steady state. Second, a dynamic model might exhibit multiplicity in response to a one-time shock of the same sort analyzed by Ball and Romer.[5] We address both possibilities.

In our analysis of steady states, we derive a best-response correspondence with a similar interpretation to that in Ball and Romer [1991]. It gives the conditional price adjustment rule that is the steady-state solution to the individual firm's problem, as a function of the common (steady-state) adjustment rule chosen by all other firms. The price adjustment rule for a typical firm takes the form of a cutoff value of the menu cost, below which a firm will choose to adjust its price. Equivalently, it can be interpreted as an *ex ante* adjustment

[2]The paper by Howitt [1981] is an important forerunner of the work of Ball and Romer. In his model, a firm can incur a fixed cost to observe the current value of an aggregate shock, in response to which it may adjust its price. Multiple equilibria can arise because of the feedback from the prices chosen by other firms who observe the shock to the value to an individual firm of observing the shock.

[3]Indeed, Ball and Romer write (pp. 538-539) that "[o]ur results suggest that coordination failure is at the root of inefficient non-neutralities of money."

[4]Our discussions of Ball and Romer [1991] will be referring specifically to their section 3.A, where heterogeneity is introduced in the form of random menu costs.

[5]There also might be multiple equilibria in terms of the timing of adjustment. In particular, models with time-dependent pricing often have both staggered equilibria, in which a constant fraction of firms adjust in each period, and synchronized equilibria, where all firms adjust at the same time. Such multiplicity could perhaps also be present under state-dependent pricing. We are investigating this possibility in other ongoing work (John and Wolman 2004).

probability.[6] As in Ball and Romer, price adjustment can exhibit complementarity in our model, implying that the best-response correspondence may slope up. However, the extent of complementarity and hence the scope for multiple steady state equilibria depend crucially on the discount factor.

When the discount factor is close to zero – so that a firm's problem is nearly static – complementarity is unambiguously present. The mechanism is that, in the face of inflation, a greater amount of aggregate adjustment raises the price level, which in turn lowers aggregate demand and the real wage. The net effect is to raise the level of a firm's profits, as well as marginal profits (with respect to a firm's adjustment probability) and hence to increase the incentive for a firm to adjust.

When the discount factor is close to one – the range relevant for business cycle analysis – we show that the model does not generate the complementarity at a fixed point typically necessary for multiple steady states. First, an increase in aggregate adjustment may decrease the price level, because it is optimal for forward-looking firms that adjust more frequently to set a lower price when they do adjust. Second, even if the net effect of higher aggregate adjustment on the level of a firm's profits is positive, the firm may choose to adjust less frequently. This is because for high values of the discount factor, firms optimally earn relatively low profits in periods when they adjust their price, meaning that expected discounted profits are decreasing in the firm's adjustment probability. The absolute magnitude of marginal profits rises with aggregate adjustment, but an individual firm responds by adjusting less frequently because its marginal profits are negative.

The same theme recurs in our investigation of transitional dynamics in response to a shock. In a one-period (static) model of the sort analyzed by Ball and Romer [1991], we show that there is complementarity in price adjustment whenever the current money supply is sufficiently high. We confirm Ball and Romer's result that multiple equilibria are possible in a static setting, although multiplicities appear to be confined to a small region of the parameter space. In a dynamic model, complementarity may still occur, but it does not occur

[6] There are of course parallels between menu cost models and other models with fixed adjustment costs. Discrete-time treatments of investment with fixed adjustment costs can be found, for example, in Cooper, Haltiwanger and Power [1999] and Thomas [2002], while Fisher and Hornstein [2000] analyze a discrete-time model of inventories with fixed costs.

universally, and in an infinite horizon model we find no examples of multiple equilibrium responses to a shock.

Less reassuringly, we find nonexistence of symmetric steady state equilibrium with pure strategies in a small region of the parameter space. When the money growth rate (and thus the inflation rate) is high enough, there is a unique symmetric steady state equilibrium in which all prices adjust every period. When the money growth rate is somewhat lower, there is a unique symmetric steady state equilibrium in which firms adjust their prices after either one or two periods, depending on the realization of their idiosyncratic fixed cost of adjustment. In an intermediate range for money growth, there is nonexistence of symmetric steady state equilibrium with pure strategies. Nonexistence is not a knife-edge result, though it is also not widespread. At a technical level, nonexistence is indicated by a discontinuity in the steady state best response correspondence. That discontinuity reflects the fact that an adjusting firm is indifferent between two strategies. One involves setting a low price and adjusting with certainty in the next period. The other involves setting a high price and adjusting in the next period only in response to a low adjustment cost. This explanation reveals that nonexistence arises because the model is set in discrete time.

The paper proceeds as follows. Section 2 describes the model in full generality. In Section 3, we turn our attention to steady states. We first derive aggregate variables conditional on firms' behavior, then describe the optimal behavior of firms. We study equilibria using the steady-state best-response correspondence that links the price-adjustment decision of a single firm to the decisions of all other firms. We prove analytically that multiple equilibria are unlikely to occur when the discount factor approaches unity, and we provide sufficient conditions to rule out multiplicity. We then use numerical methods to confirm this finding; our computations show that multiplicity may occur for small values of the discount factor, but we find no evidence of multiplicity for plausible values of the discount factor. The numerical results also illustrate the possibility of nonexistence. Section 4 solves for the model's transitional dynamics and uses this solution to study the kind of one-time shock considered by Ball and Romer: we find a unique equilibrium in response to such a shock. Section 5 concludes.

2. The Model

Our model is a simplified version of that in Dotsey, King and Wolman [1999]. Nominal rigidity is introduced into a monopolistic competition framework through the assumption of fixed costs of price adjustment, as in Blanchard and Kiyotaki [1987]. The size of the state space is limited by the number of different prices charged, and positive inflation together with bounded costs means that a finite number of prices are charged. In the version of the model that we study, the money supply is the only fundamental exogenous variable.

2.1. Consumers

An infinitely-lived representative consumer has preferences over consumption (c_t) and labor (n_t), where consumption is a nonlinear CES aggregate of differentiated products:

$$c_t = \left[\int_0^1 c_t(z)^{(\varepsilon-1)/\varepsilon} dz\right]^{\varepsilon/(\varepsilon-1)}, \varepsilon > 1. \tag{2.1}$$

The consumer maximizes the expected present discounted value of utility,

$$E\left(\sum_{j=0}^{\infty} \beta^j u(c_{t+j}, n_{t+j})\right), \tag{2.2}$$

where $u(c, n)$ is a standard utility function. The consumer sells labor in a competitive labor market for the real wage (w_t), and also receives dividend payments from firms ($\tilde{\Pi}_t$).[7] There is no vehicle for saving, so from the consumer's perspective the model presents a sequence of static problems in which all income is used up each period purchasing consumption goods from firms:[8]

$$c_t = w_t n_t + \tilde{\Pi}_t. \tag{2.3}$$

We introduce money demand in a simple way, by assuming that velocity is unity. Thus,

$$\int_0^1 P_t(z) c_t(z) dz = M_t, \tag{2.4}$$

[7]We could easily amend our model to include fixed overhead costs that would absorb the profits, as in Hornstein [1993] for example. Our basic conclusions would not be affected in any way. In particular, the results would be identical in our benchmark case, because the labor supply decision in that case is independent of the level of profits.

[8]It is straightforward to allow for capital accumulation in this model (see Dotsey, King and Wolman [1997]). We study a model without capital in order to keep the analysis as tractable as possible.

where $P_t(z)$ is the nominal price of good z and M_t is the aggregate money stock. (Our main findings are essentially unaltered if we instead derive money demand from a cash-in-advance constraint; see Section 3.6.)

2.2. Firms

Each firm, z, possesses a linear technology for producing output $(y_t(z))$:

$$y_t(z) = n_t^y(z) = (n_t(z) - n_t^p(z)), \tag{2.5}$$

where $n_t(z)$ is total labor employed by the firm, $n_t^y(z)$ is labor utilized in production, and $n_t^p(z)$ is labor employed in price adjustment. Given the preferences specified for the consumer, each firm faces a demand curve

$$c_t(z) = \left(\frac{P_t(z)}{P_t}\right)^{-\varepsilon} \cdot c_t, \tag{2.6}$$

where

$$P_t = \left[\int_0^1 P_t(z)^{1-\varepsilon} dz\right]^{1/(1-\varepsilon)}. \tag{2.7}$$

Firms meet all demand at the price they set, so

$$y_t(z) = c_t(z) \ \forall z. \tag{2.8}$$

Each period, an individual firm has a cost in labor units of changing its price, ξ, which is an *independent* draw (across firms and across time) from a continuous distribution, $F(\xi)$, on $[0, B]$.[9] Upon observing its cost of changing prices, each firm either leaves its price at the level in effect in the previous period, or adjusts to its optimal price. If the firm adjusts its price, it incurs a cost $w\xi$ denominated in units of output, because it must hire ξ units of labor at the real wage w. It will be convenient to normalize the nominal price with which an individual firm enters period t by the nominal money supply in period $t-1$. We denote this normalized price by $x_t(z) = P_{t-1}(z)/M_{t-1}$. Notice that the price with which the firm enters, relative to the general price level in period t, equals

$$\frac{P_{t-1}(z)}{P_t} = \frac{P_{t-1}(z)}{M_{t-1}} \cdot \frac{M_{t-1}}{M_t} \cdot \frac{M_t}{P_t} = \frac{x_t(z)c_t}{\mu_t},$$

[9]Heterogenity in menu costs is desirable for two reasons. First, it enabled the model to match qualitatively the observation that, while price changes are infrequent, they also vary widely in magnitude. Second, heterogeneity allows equilibrium to be determined without randomization (in most cases).

where μ_t is the gross money supply growth rate between $t-1$ and t ($\mu_t \equiv M_t/M_{t-1}$), and where we use the fact that $c_t = M_t/P_t$. The advantage of this normalization is that it allows us to define a recursive equilibrium.

Let s denote the aggregate state of the economy; s comprises exogenous variables and predetermined endogenous variables sufficient to determine current period equilibrium values of all endogenous variables. The problem facing firm z, who enters the period with normalized price $x(z)$ and draws an adjustment cost ξ is

$$v\left(x(z),\xi;s\right) = \max \begin{cases} H(x(z);s) \equiv \lambda \cdot \pi\left(\frac{x(z)}{\mu};s\right) + \beta E v\left(\frac{x(z)}{\mu},\xi';s'|s\right), \\ \\ Q(\xi;s) \equiv -\lambda \cdot w \cdot \xi + \max_{x_0}\left[\lambda \cdot \pi\left(x_0;s\right) + \beta E v\left(x_0,\xi';s'|s\right)\right] \end{cases} \tag{2.9}$$

Because individuals own the firms, a unit of real profits is valued in utility terms at λ, the representative agent's marginal utility of consumption. In (2.9), $H\left(x(z);s\right)$ is the value of not adjusting, $Q\left(\xi;s\right)$ is the value of adjusting, and the function $\pi\left(.\right)$ is the firm's contemporaneous profits as a function of its price normalized by the money stock, $\tilde{x}$:

$$\pi\left(\tilde{x};s\right) \equiv \left[\tilde{x} \cdot c - w\right] \cdot c^{1-\varepsilon}\left(\tilde{x}\right)^{-\varepsilon} \tag{2.10}$$

Our expression for profits makes use of our normalization of prices together with money-market equilibrium ($c = M/P$). It is straightforward to verify Blackwell's sufficient conditions for the mapping in (2.9). Therefore, there is a unique bounded function $v\left(x\left(z\right),\xi;s\right)$ satisfying (2.9).[10]

Notice that the value of adjusting is independent of the firm's normalized price and is strictly decreasing in the cost a firm draws ($\partial Q/\partial\xi < 0$), while the value of not adjusting is independent of the cost the firm draws. It follows that the optimal adjustment policy involves a threshold cost: for any predetermined normalized price ($x(z)$) and aggregate state (s), there exists a unique cost $\xi^c\left(x(z),s\right)$ such that firm z with price $x(z)$ facing aggregate state s will adjust if and only if the cost it draws is less than $\xi^c\left(x(z),s\right)$:

$$v\left(x(z),\xi;s\right) = \begin{cases} H\left(x(z);s\right), \text{ for } \xi > \xi^c\left(x(z),s\right) \\ Q\left(\xi;s\right), \text{ for } \xi \le \xi^c\left(x(z),s\right). \end{cases} \tag{2.11}$$

[10] The function is bounded above by the value of obtaining profit-maximizing profits each period and adjusting costlessly; it is bounded below by the value of obtaining profit-maximizing profits each period and adjusting with the highest cost of adjustment.

The *ex ante* probability that a firm entering the period with normalized price x will choose to adjust is given by $\alpha = F\left(\xi^c\left(x(z), s\right)\right)$. There is an equivalence between the cutoff cost ξ^c and the adjustment probability α; for our analysis it is easier to work in terms of the adjustment probability. Further, because the future looks the same to all firms that choose to adjust, such firms will all choose the same new price.

2.3. Monetary Policy

The model is closed by adding a policy rule for the monetary authority. We assume that the monetary authority simply picks an exogenous process for the money growth rate μ_t. In section 3 money growth is constant ($\mu_t = \mu \ \forall \ t$), whereas in Section 4 we consider a transitory shock to money growth: $\mu_t = \mu$ for $t = 1, 2, ...$, but $\mu_0 \neq \mu$.

2.4. The aggregate state

Let $\boldsymbol{\omega}_t$ be the $J-$element vector containing the fractions of firms in period $t-1$ that charged a price set in periods $t-1, t-2, t-3, ...t-J$. And, let $\mathbf{x}_t$ be the vector of prices that these firms charged in period $t-1$, normalized by the money supply in period $t-1$. The aggregate state s_t is given by $\{\boldsymbol{\omega}_t, \mathbf{x}_t\}$. We will focus on the region of the parameter space containing only equilibria in which $J \leq 2$, so that firms go at most two periods without adjusting their price. In general, high enough money growth or low enough costs of price adjustment will guarantee this:

Assumption 1 **($J \leq 2$).** The maximum cost of price adjustment is low enough, and the inflation rate is high enough, that in equilibrium no firm chooses to go more than two periods with the same price.

Comment The assumption implies that $\boldsymbol{\omega}_t$ is a scalar, which we will denote ω_t. The assumption likewise implies that $\mathbf{x}_t$ is a scalar, denoted x_t, since the only price that is relevant is the price that was set by firms who adjusted last period. Ideally, this assumption would be framed explicitly in terms of parameters – for example values of B and μ such that no firm would choose to go more than two periods without adjusting its price. While it is straightforward to verify the assumption for particular examples, it is difficult – and unnecessary for our purposes – to write a general condition in terms of the parameters of the model. Instead, we proceed as if there is a suitable underlying assumption on the parameters. In our numerical analysis of steady states, we will verify that Assumption 1 holds for each case we analyze. For a sampling of cases where the assumption does not hold, we extend our numerical analysis accordingly.

The distribution of firms evolves according to

$$\omega_{t+1} = (1 - \omega_t) + \alpha_t \omega_t. \tag{2.12}$$

The fraction ω_{t+1} of firms that adjust their price in the current period comprises all of the firms that did not adjust in the previous period, plus a fraction α_t of those firms that did adjust in the previous period.

2.5. Recursive Equilibrium

Here we define a recursive equilibrium. In subsequent sections we investigate steady-state equilibria and the equilibrium response to a monetary shock. It will be clear how to view steady-state equilibrium as recursive. For the response to a monetary shock, we will be explicitly concerned with the possibility of multiple equilibria. While the standard notion of a recursive equilibrium embodies uniqueness, we can study the possibility of multiple equilibrium responses to a shock by expanding the state vector to include an additional nonfundamental state variable.

Before defining an equilibrium, it is useful to rewrite the price index (2.7), under the assumption that only two prices are charged (that is, $J = 2$). In this case, letting P_t^* denote the nominal price chosen by an adjusting firm in period t, (2.7) implies

$$P_t = \left[\omega_{t+1} \left(P_t^*\right)^{1-\varepsilon} + (1 - \omega_{t+1}) \left(P_{t-1}^*\right)^{1-\varepsilon}\right]^{1/(1-\varepsilon)},$$

Written in terms of normalized prices ($x_t = P_{t-1}^*/M_{t-1}$), this becomes

$$P_t = \left[\omega_{t+1} \left(x_{t+1} M_t\right)^{1-\varepsilon} + (1 - \omega_{t+1}) \left(x_t M_{t-1}\right)^{1-\varepsilon}\right]^{1/(1-\varepsilon)}.$$

Now recall that the normalized price index is simply the inverse of real balances and hence, from the money demand equation, the inverse of consumption. That is $P_t/M_t = 1/c_t$. Using the superscript prime ($'$) to denote the value of a variable in period $t+1$, we have

$$\begin{aligned} 1 &= \omega' \left(cx'\right)^{1-\varepsilon} + \left(1-\omega'\right)\left(cx/\mu\right)^{1-\varepsilon} \qquad (2.13) \\ &\Rightarrow \; c^{\varepsilon-1} = \omega' \left(x'\right)^{1-\varepsilon} + \left(1-\omega'\right)\left(x/\mu\right)^{1-\varepsilon} \\ &\Rightarrow \; c = \left[\omega' \left(x'\right)^{1-\varepsilon} + \left(1-\omega'\right)\left(x/\mu\right)^{1-\varepsilon}\right]^{\frac{1}{\varepsilon-1}}. \end{aligned}$$

A fraction ω' of firms adjust this period and set the normalized price x'; the remainder set the price x in the previous period, and money growth reduces this in normalized terms to x/μ. The distribution of normalized prices thus pins down aggregate demand. The logic behind this derivation is the following. The price index determines the price level in a given period as a function of the prices charged by all firms. Because we normalize individual prices by the money supply, the price level can be written as a function of the normalized prices and of the money supply. And because aggregate demand equals the real money supply, we can solve for aggregate demand as a function of the normalized prices.

A recursive equilibrium is constructed as follows. Suppose that firms with a one-period old price choose to adjust with probability α and that adjusting firms set the price x'. Given ω, we can find ω' from (2.12), and thus we can derive c from (2.13). Because we know the prices charged by all firms, we can then use the technology to obtain the labor input in production.[11] We can also calculate labor used in price adjustment: firms that adjust for sure this period have an average menu cost equal to the unconditional mean of the distribution of costs ($E(\xi)$), while firms that choose to adjust have an average menu cost equal to the mean conditional on drawing a cost below the cutoff, which equals $E\left(\xi|\xi < F^{-1}(\alpha)\right)$. Labor market clearing implies that this total labor input must equal labor supply, and so we finally use the first-order condition from the consumer's problem to pin down the equilibrium real wage. Thus we can solve for aggregate variables as a function of arbitrary behavior by firms. We complete our definition by specifying optimal behavior for firms and requiring that this behavior be consistent with the assumed behavior of firms. Our formal definition follows.

Definition 2.1. *A **recursive equilibrium** in which no firm goes more than two periods*

[11] Consider an adjusting firm. This firm sets the normalized price x'. The demand curve implies that the output of this firm equals $\left(cx'\right)^{-\varepsilon} c = c^{1-\varepsilon} x^{-\varepsilon}$. The linear technology implies that this is also the solution for labor input. A similar calculation allows us to derive labor input for non-adjusting firms, and total labor input is just the appropriate weighted sum.

without changing its price is a pair of functions $\alpha(s)$, $x^*(s)$ satisfying the conditions listed below, where $s = \{\omega, x\}$. The function $\alpha(s)$ determines the current adjustment probability α for firms who changed their price in the previous period, in terms of the state, and the function $x^*(s)$ determines the price set by adjusting firms (x') in terms of the state. The conditions which must hold for these functions to represent an equilibrium are that

1. *The current fraction ω' of adjusting firms is determined by the law of motion (2.12), given $\alpha = \alpha(s)$.*

2. *Consumption (c) is determined by (2.13), given $x' = x^*(s)$.*

3. *Labor input used in goods production (n^y) is determined by*

$$n^y = c^{1-\varepsilon} \cdot \left(\omega' \cdot (x')^{-\varepsilon} + (1-\omega') \cdot (x/\mu)^{-\varepsilon}\right). \tag{2.14}$$

4. *Labor input used in price adjustment (n^p) is determined by*

$$n^p = \alpha\omega' E\left(\xi | \xi < F^{-1}(\alpha)\right) + (1-\omega') E(\xi). \tag{2.15}$$

5. *Total labor supplied is the sum of labor used in these two activities*

$$n = n^y + n^p. \tag{2.16}$$

6. $\lambda = u_c(c, n)$

7. *The real wage w is consistent with optimal labor supply:*

$$w = -u_n(c,n)/u_c(c,n). \tag{2.17}$$

8. *Given that aggregate variables are determined by s according to 1-7, the functions $\alpha(s)$, $x^*(s)$ represent optimal behavior by firms:*

   a. *An individual firm's optimal price is consistent with aggregate pricing:*

$$x^*(s) = \arg\max_{x_0} \left[\lambda(s) \cdot \pi(x_0; s) + \beta E v(x_0, \xi'; s')\right], \tag{2.18}$$

where

$$v(q,\xi;s)=\max\left\{\begin{array}{c}\lambda(s)\cdot\pi(q/\mu;s)+\beta Ev(q/\mu,\xi';s')\\ -\lambda(s)\cdot w(s)\cdot\xi+\max_{x_0}\left[\lambda(s)\cdot\pi(x_0;s)+\beta Ev(x_0,\xi';s')\right]\end{array}\right\}. \tag{2.19}$$

b. *An individual firm's optimal adjustment rule is consistent with the aggregate adjustment rule:*

$$\begin{aligned}&w(s)\lambda(s)F^{-1}(\alpha(s)) \qquad (2.20)\\ \leq\ &\lambda(s)\cdot\pi(x^*(s);s)+\beta Ev(x^*(s),\xi';s')-\lambda(s)\cdot\pi(x/\mu;s)-\beta Ev(x/\mu,\xi';s'),\end{aligned}$$

with equality if $\alpha(s)<1$. In this definition, the left-hand side is the adjustment cost associated with the ex-ante adjustment probability $\alpha(s)$. The right-hand side is the change in value associated with shifting from the pre-existing price x (that is, the price set by a representative firm that enters the period with a price that is one period old) to the optimal price $x^*(s)$. Thus this expression implies that a representative firm chooses $\alpha=\alpha(s)$ when the state is s.

3. Steady-State Equilibrium

In this section we characterize steady-state equilibrium with inflation, and so assume $\mu_t=\mu>1$, $\forall\, t$.[12] We derive equilibrium in the manner just described: we first take as given the decision rules of firms, and solve for aggregate variables. We then consider the behavior of an individual firm that faces a constant aggregate state. It faces a price-adjustment decision (should it leave its price unchanged, or incur the menu cost that allows it to adjust?), together with a price-setting decision (if it does adjust, what price should it choose?). We solve for optimal price-setting given an arbitrary adjustment pattern, and then we characterize the condition for optimal adjustment.[13] We write this condition as the best-response of an

[12]The presence of inflation implies that the monetary authority is obtaining seigniorage revenue. We do not include an explicit government budget constraint; hence we are implicitly assuming that seigniorage revenues are returned to the consumers in the form of lump-sum transfers.

[13]The firm's choice of price naturally affects its future adjustment decisions, but at an optimum, its price will be optimal treating the pattern of adjustment as exogenous.

individual firm to the adjustment decisions of other firms. Steady state equilibria are fixed points of the best-response correspondence.

3.1. Aggregate Variables Given Firms' Behavior

We first determine the aggregate variables (w, c, λ) given arbitrary symmetric behavior by firms, in a steady state. All firms that adjusted in the previous period adjust in the current period with probability $\bar{\alpha}$, and all adjusting firms set the price $\bar{x}$. We then proceed according to the definition of recursive equilibrium above.

As a preliminary, we define the following functions, which will prove useful in our analysis[14]

$$r(\alpha;\beta) \equiv \frac{1+\beta(1-\alpha)\mu^{\varepsilon-1}}{1+\beta(1-\alpha)} \geq 1; \tag{3.1}$$

$$g(\alpha;\beta) \equiv \frac{1+\beta(1-\alpha)\mu^{\varepsilon}}{1+\beta(1-\alpha)\mu^{\varepsilon-1}} \geq 1. \tag{3.2}$$

Versions of these functions will represent the normalized price index and optimal pricing, respectively.

3.1.1. The Distribution of Firms, the Price Index, and Aggregate Demand

In steady state the distribution of firms is constant with respect to when they last set their price, which implies that the fraction of firms adjusting their price is

$$\omega' = \frac{1}{2-\bar{\alpha}}, \tag{3.3}$$

and the fraction of firms not adjusting their price is $1-\omega' = (1-\bar{\alpha})/(2-\bar{\alpha})$. Because $x = x' = \bar{x}$, (2.13) implies,

$$c^{\varepsilon-1} = \omega'(\bar{x})^{1-\varepsilon} + (1-\omega')(\bar{x}/\mu)^{1-\varepsilon}, \tag{3.4}$$

which allows us to solve for aggregate demand as a function of firms' pricing behavior:

$$\begin{aligned} c(\bar{x},\bar{\alpha}) &= (\bar{x})^{-1} \cdot \left[\frac{1+(1-\bar{\alpha})\mu^{\varepsilon-1}}{1+(1-\bar{\alpha})}\right]^{1/(\varepsilon-1)} \\ &= (\bar{x})^{-1} \cdot r(\bar{\alpha};1)^{1/(\varepsilon-1)}. \end{aligned} \tag{3.5}$$

[14] To avoid notational clutter we suppress the dependence of $r(\cdot)$ and $g(\cdot)$ on the other parameters.

It is easy to show that $\partial r()/\partial\bar{\alpha} < 0$, which means that an increase in $\bar{\alpha}$ is associated, in equilibrium, with a decrease in $\bar{x}c(\bar{x},\bar{\alpha})$. When more firms adjust, the price level is closer to the price chosen by adjusting firms, and so the relative price $\bar{x}c$ of adjusting firms is lower. The price level is growing at rate μ, so holding fixed $\bar{x}$, an increase in the probability of adjustment implies a higher price level and lower aggregate demand.

3.1.2. Labor Input

Labor input used in goods production (n^y) is determined by

$$\begin{aligned} n^y(\bar{x},\bar{\alpha}) &= c(\bar{x},\bar{\alpha})^{1-\varepsilon}\cdot[\omega'+(1-\omega')\cdot\mu^{\varepsilon}](\bar{x})^{-\varepsilon} && (3.6)\\ &= c(\bar{x},\bar{\alpha})^{1-\varepsilon}\cdot(\bar{x})^{-\varepsilon}\cdot\frac{g(\bar{\alpha};1)}{r(\bar{\alpha};1)}, && (3.7)\end{aligned}$$

and labor input used in price adjustment (n^p) is determined by

$$n^p(\bar{\alpha}) = \frac{\bar{\alpha}E\left(\xi|\xi<F^{-1}(\bar{\alpha})\right)+(1-\bar{\alpha})E(\xi)}{2-\bar{\alpha}}. \tag{3.8}$$

Total labor supplied is the sum of labor used in these two activities,

$$n(\bar{x},\bar{\alpha}) = n^y(\bar{x},\bar{\alpha})+n^p(\bar{\alpha}). \tag{3.9}$$

3.1.3. The Real Wage and the Marginal Utility of Income

Utility maximization by consumers implies

$$\lambda(\bar{x},\bar{\alpha}) = u_c\left(c(\bar{x},\bar{\alpha}),n(\bar{x},\bar{\alpha})\right). \tag{3.10}$$

Finally, optimal labor supply implies

$$w(\bar{x},\bar{\alpha}) = -\frac{u_n\left(c(\bar{x},\bar{\alpha}),n(\bar{x},\bar{\alpha})\right)}{\lambda(\bar{x},\bar{\alpha})}. \tag{3.11}$$

We have thus solved for all aggregates given arbitrary behavior by firms.

3.2. Derivation of Firms' Behavior

We now solve for the optimal behavior of an individual firm, taking as given the aggregate steady-state variables w, c, and λ. (These variables depend upon the state, s; we now suppress that dependence for notational convenience.) The functional equation describing the firm's problem is given by (2.9) with $s' = s$. If the firm adjusts, it sets a new price that solves the subproblem, $\max_{x_0} \{\lambda\pi(x_0; s) + \beta Ev(x_0, \xi'; s)\}$. Henceforth we will denote the maximized value of this subproblem by $\mathsf{v}(s)$:

$$\mathsf{v}(s) \equiv \max_{x_0} [\lambda\pi(x_0; s) + \beta Ev(x_0, \xi'; s)]. \tag{3.12}$$

Thus $Q(\xi; s) = -\lambda w\xi + \mathsf{v}(s)$. Let x^* denote a solution to this maximization problem.

Because of the discrete adjustment decision, the firm's value function is not concave, and because of discrete time x^* need not be unique, although of course the maximized value of the subproblem is unique. Nonuniqueness (and associated non-quasiconcavity of the value function) arises because, for a given adjustment cost, a firm's value as a function of its incoming price has points of non-differentiability where the firm would switch from, for example, a maximum of one period without adjustment to a maximum of two periods without adustment. There may be local maxima on both sides of such a point of non-differentiability.

For the remainder of the paper, the analysis is complicated by the need to accommodate possible nonuniqueness of x^*. Nonuniqueness of x^* is an entirely different matter from nonuniqueness of steady-state equilibrium. The former is a characteristic of an individual firm's optimal policy, and is neither necessary nor sufficient for the latter. In fact, we will see below that nonuniqueness of x^* gives rise in some circumstances not to multiplicity of equilibrium, but to non-existence of equilibrium.

We determine optimal firm behavior in two steps. In the first step a firm chooses the optimal price, taking α as given, and in the second step it chooses the optimal α. Notice that, in this analysis, α is the adjustment strategy chosen by the *individual* firm, and is distinct from the *aggregate* adjustment strategy $\bar{\alpha}$ being chosen by all other firms. Because the aggregate state is constant, it is optimal for a firm to choose a constant α.

3.2.1. Optimal Pricing, Conditional on Adjustment

Abusing notation somewhat, we define $\mathsf{v}(\alpha; s)$ to be the value gross of adjustment cost of an adjusting firm which is following the *arbitrary* constant adjustment policy α but setting the optimal price x^* that is associated with the policy. Because firms never go more than two periods with the same price, we can write $\mathsf{v}(\alpha;s)$ using (3.12) as follows:

$$\begin{aligned}\mathsf{v}(\alpha;s) &= \lambda\pi(x^*(\alpha)) + \beta\alpha\left(\mathsf{v}(\alpha;s) - w\lambda E\left(\xi|\xi < F^{-1}(\alpha)\right)\right) + \\ &\quad \beta(1-\alpha)\left[\lambda\pi\left(\frac{x^*}{\mu}\right) + \beta\left(\mathsf{v}(\alpha;s) - w\lambda E(\xi)\right)\right].\end{aligned} \tag{3.13}$$

The present value of a firm that has just adjusted its price equals the sum of expected future profits and costs; $\mathsf{v}(\alpha;s)$ appears on the right hand side because the firm will eventually adjust, whereupon it will have the same value as at the current date. Solving for $\mathsf{v}(\alpha;s)$, we obtain[15]

$$\mathsf{v}(\alpha;s) = \frac{\lambda\pi(x^*) - \beta\alpha w\lambda E\left(\xi|\xi < F^{-1}(\alpha)\right) + \beta(1-\alpha)\left[\lambda\pi\left(\frac{x^*}{\mu}\right) - \beta w\lambda E(\xi)\right]}{(1-\beta)(1+\beta(1-\alpha))}. \tag{3.14}$$

Lemma 3.1. *Given α, the optimal price chosen by an adjusting firm satisfies*

$$\frac{\partial\pi(x^*)}{\partial x^*} + \beta(1-\alpha)\frac{\partial\pi\left(\frac{x^*}{\mu}\right)}{\partial x^*} = 0 \tag{3.15}$$

Proof: The proof is immediate from (3.14). The optimal price is precisely the x^* that ensures $\mathsf{v}(\alpha;s)$ is maximized. The cost terms do not depend upon x, and the result follows. □

It is now easy to derive the price chosen by an adjusting firm as a function of its adjustment pattern. Recall that profits are given by $\pi(x) = [x\cdot c - w]\cdot c^{1-\varepsilon}x^{-\varepsilon}$, which implies

$$\frac{\partial\pi(x^*)}{\partial x^*} = c^{1-\varepsilon}(x^*)^{-\varepsilon-1}\left[(1-\varepsilon)cx^* + \varepsilon w\right]; \tag{3.16}$$

$$\frac{\partial\pi\left(\frac{x^*}{\mu}\right)}{\partial x^*} = c^{1-\varepsilon}(x^*)^{-\varepsilon-1}\left[(1-\varepsilon)cx^*\mu^{\varepsilon-1} + \varepsilon w\mu^{\varepsilon}\right]. \tag{3.17}$$

[15] Two examples provide intuition for this expression. First, consider a firm that adjusts its price every period. For this firm $\alpha = 1$, and

$$\mathsf{v}(1;s) = \frac{\lambda\pi(x^*) - \beta w\lambda E(\xi)}{1-\beta},$$

which is simply the present discounted value of the profits from the optimal price minus the average menu cost. Second, consider a firm that adjusts with certainty every other period, so that $\alpha = 0$. Then

$$\mathsf{v}(0;s) = \frac{\lambda\pi(x^*) + \beta\lambda\pi\left(\frac{x^*}{\mu}\right) - \beta^2 w\lambda E(\xi)}{1-\beta^2}.$$

In this case the firm receives profits equal to $\pi(x^*)$ and $\pi(x^*/\mu)$ in alternating periods. Every other period it incurs a menu cost whose expected value is the unconditional mean of the distribution. This two-period flow of profits and costs is discounted at the two-period discount rate β^2.

Substituting (3.16) and (3.17) into (3.15), the optimal normalized price satisfies

$$x^{*}(\alpha;s)=\left(\frac{\varepsilon}{\varepsilon-1}\right)\cdot g(\alpha;\beta)\cdot\frac{w}{c}, \tag{3.18}$$

Real marginal cost in this model equals the real wage, so nominal marginal cost normalized by the money supply equals w/c. In a flexible price model ($\alpha = 1$), the markup of an adjusting firm's normalized price over marginal cost is $(\varepsilon/(\varepsilon-1))$. This is also the value of the markup in a steady state with complete discounting ($\beta = 0$) or with a constant price level ($\mu = 1$), since in either case $g(\cdot) = 1$. As long as $\beta > 0$, $\mu > 1$, and $\alpha < 1$, however, it follows that $g(\alpha;\beta) > 1$: an adjusting firm's markup exceeds the static markup. Firms set their price knowing that this price may be in effect not just in the current period, but also in subsequent periods when the general price level will be higher.

The markup charged by an adjusting firm is increasing in β ($\partial g/\partial\beta > 0$), because firms then place relatively more weight on future periods (when inflation will have eroded the price that they set). The markup is decreasing in the price adjustment probability α ($\partial g/\partial\alpha < 0$), because the probability of *not* adjusting plays the same role as the discount factor. Finally, it is easy to see that the markup of an adjusting firm is also increasing in the inflation rate: higher values of μ induce firms to set a higher markup.

3.2.2. Optimal Adjustment

A necessary and sufficient condition for optimality is that the firm is maximizing its value:

$$\alpha(s)=\arg\max \mathsf{v}(\alpha;s). \tag{3.19}$$

When α is an optimal policy, $\mathsf{v}(\alpha;s)$ will equal $\mathsf{v}(s)$. It is tempting to think that optimal adjustment is characterized completely by the condition that a firm's adjustment probability equal the probability of drawing a cost smaller than the gain from adjusting. That condition is not sufficient because (a), it can yield a local minimum (associated with a solution to the first order condition for (3.19)), and (b), as we will see below, $\mathsf{v}(\alpha;s)$ may have more than one *local* maximum.[16]

[16] The condition which balances the probability of adjustment against the probability of drawing a cost less than the gain from adjusting is described in more detail on page 21 of an early version of this paper, available at http://www.rich.frb.org/pubs/wpapers/pdfs/wp99-5.pdf

3.3. Equilibrium

Definition 3.2. *For a given constant rate of money growth μ, a* ***symmetric steady-state equilibrium*** *is a recursive equilibrium as defined above, together with a state $\hat{s} = \{\hat{x}, \hat{\omega}\}$, such that $x^* (\alpha (\hat{s}) ; \hat{s}) = \hat{x}$ and $\hat{\omega} = (1 - \hat{\omega}) + \alpha (\hat{s}) \hat{\omega}$.*

The first condition simply states that, when faced with the state $\hat{s}$, firms find it optimal to set the price $\hat{x}$, while the second condition ensures that, when firms choose their optimal adjustment policy, the distribution of firms replicates itself through time.

We analyze equilibrium in the model through the mapping from aggregate adjustment patterns to the adjustment pattern of an individual firm. We thus assume that all other firms are choosing the arbitrary adjustment pattern $\bar{\alpha}$, and that these firms choose the optimal price associated with that adjustment pattern, $x^* (\bar{\alpha})$ (see 3.18). By substituting the expressions for aggregate demand (3.5) and the real wage (3.11) for given firm behavior into the optimal pricing equation (interpreted now as holding for all firms) we arrive at an equation that implicitly defines the normalized price charged by adjusting firms, x^*, as a function of the adjustment probability $\bar{\alpha}$:

$$x^* = -\left(\frac{\varepsilon}{\varepsilon - 1}\right) \cdot \frac{u_n (c (x^*, \bar{\alpha}), n (x^*, \bar{\alpha}))}{\lambda (x^*, \bar{\alpha}) \cdot c (x^*, \bar{\alpha})} \cdot g (\bar{\alpha}; \beta) \tag{3.20}$$

That is, this expression gives us the general equilibrium relationship between x^* and $\bar{\alpha}$. It should be distinguished from (3.18) which gives the solution for $x^* (\alpha)$ for an individual firm, taking the state as given. Our solution for x^* can then be eliminated from (3.5) and (3.11), so that aggregate demand and the real wage are expressed as functions of only the adjustment probability $\bar{\alpha}$. At this point we have solved for the state, s, as a function of $\bar{\alpha}$. We can then find the best response of an individual firm to this adjustment behavior of other firms (the solution to 3.19). A symmetric steady-state equilibrium is a fixed point of this steady-state best-response correspondence, and any fixed point of the steady-state best-response correspondence is a symmetric steady-state equilibrium.

3.4. The Best-Response Correspondence: An Analytical Approach

Specializing to a particular form of the utility function allows us to study the best response correspondence analytically (but see Section 3.6 for sensitivity analysis).

Assumption 2 The instantaneous utility function takes the form:

$$u(c,n) = \ln c - \chi n. \tag{3.21}$$

Comment. This specification (which is common in business cycle research) implies that the labor supply elasticity is infinite. The advantage of this specification is tractability: it makes aggregate demand proportional to the real wage.[17] Specifically, (3.11) becomes simply $w = \chi c$. With more general preferences, the real wage depends upon the adjustment probability, because changes in the adjustment probability imply shifts in labor demand.

3.4.1. Preliminaries

Our analysis of the properties of the best response correspondence requires us to derive several preliminary expressions. Given $w = \chi c$, (3.20) becomes

$$x^*(\alpha) = \left(\frac{\varepsilon}{\varepsilon - 1}\right) \cdot \chi \cdot g(\alpha; \beta); \tag{3.22}$$

the optimal normalized price for an individual firm is independent of the state (and therefore independent of $\bar{\alpha}$). Also, we can eliminate w from the profit function (2.10), implying that profits are proportional to $c^{2-\varepsilon}$:

$$\pi(x^*; s) \equiv [x^* - \chi] \cdot c^{2-\varepsilon} (x^*)^{-\varepsilon}. \tag{3.23}$$

When $\varepsilon > 2$, increases in aggregate demand thus reduce profits, or – more precisely – increases in aggregate demand are associated with proportional increases in the real wage, and the net effect is to reduce profits. Recall, however, that profits are valued according to the marginal utility of consumption $\lambda = c^{-1}$, implying that increases in aggregate demand unambiguously reduce profits in utility terms:

$$\lambda(\bar{\alpha}) \pi(x^*; s) \equiv [x^* - \chi] \cdot c^{1-\varepsilon} (x^*)^{-\varepsilon}.$$

[17] Other assumptions would also generate this implication. For example, if we included overhead labor so that firms earned zero profits, then we would obtain the same results for any utility function that is logarithmic in consumption and separable in consumption and labor supply.

Using our results thus far, we can rewrite the steady-state value function (3.14) so as to make clear how individual and aggregate adjustment affect a firm's value:

$$\begin{aligned}
\mathsf{v}(\alpha; s(\bar{\alpha})) &= \frac{\lambda(\bar{\alpha})\,\pi(x^*(\alpha); s(\bar{\alpha})) - \beta\alpha w(\bar{\alpha})\,\lambda(\bar{\alpha})\,E(\xi|\xi < F^{-1}(\alpha))}{(1-\beta)(1+\beta(1-\alpha))} \\
&\quad + \frac{\beta(1-\alpha)\left[\lambda(\bar{\alpha})\,\pi\left(\frac{x^*(\alpha)}{\mu}; s(\bar{\alpha})\right) - \beta w(\bar{\alpha})\,\lambda(\bar{\alpha})\,E(\xi)\right]}{(1-\beta)(1+\beta(1-\alpha))}. \qquad (3.24)
\end{aligned}$$

Because adjustment costs are incurred as labor, the menu cost must be multiplied by the wage. But costs must also be valued according to the marginal utility of consumption. Given our specification of preferences, these two effects cancel: $(w(\bar{\alpha})\,\lambda(\bar{\alpha}) = \chi)$, and costs measured in terms of utility turn out to be independent of aggregate demand.

Gathering the profit terms and the cost terms:

$$\begin{aligned}
\mathsf{v}(\alpha; s(\bar{\alpha})) &= \frac{\pi_{SUM}(\alpha, \bar{\alpha}) - C_{SUM}(\alpha)}{(1-\beta)}, \text{ where} && (3.25) \\
\pi_{SUM}(\alpha, s(\bar{\alpha})) &\equiv \frac{\lambda(\bar{\alpha})\left[\pi(x^*(\alpha); s(\bar{\alpha})) + \beta(1-\alpha)\,\pi\left(\frac{x^*(\alpha)}{\mu}; s(\bar{\alpha})\right)\right]}{(1+\beta(1-\alpha))} \text{ and} && (3.26) \\
C_{SUM}(\alpha) &\equiv \frac{\beta\chi\left[\alpha E(\xi|\xi < F^{-1}(\alpha)) + \beta(1-\alpha)\,E(\xi)\right]}{(1+\beta(1-\alpha))}. && (3.27)
\end{aligned}$$

After some manipulation, we can write

$$\pi_{SUM}(\alpha, s(\bar{\alpha})) = \left(\frac{1}{\varepsilon}\right)\left(\frac{D(\alpha,\beta)}{D(\bar{\alpha},\beta)}\right)^{\varepsilon-1}\left(\frac{r(\alpha;\beta)}{r(\alpha;1)}\right). \qquad (3.28)$$

where we define the *aggregate demand function* $D(\alpha, \beta)$ as follows:

$$D(\alpha,\beta) \equiv \left(\frac{\varepsilon-1}{\varepsilon\chi}\right) \cdot \frac{r(\alpha;1)^{1/(\varepsilon-1)}}{g(\alpha;\beta)}.$$

We refer to this as the aggregate demand function because it equals aggregate demand when evaluated at $\alpha = \bar{\alpha}$:[18] that is,

$$c(\bar{\alpha}) \equiv \left(\frac{\varepsilon-1}{\varepsilon\chi}\right) \cdot \frac{r(\bar{\alpha};1)^{1/(\varepsilon-1)}}{g(\bar{\alpha};\beta)} \qquad (3.29)$$

We establish some properties of the aggregate demand function in the following lemma.

[18] This can be verified by referring to (3.5) and (3.22). Of course, in general equilibrium, aggregate demand is also related to the wage; indeed, in this model, they are proportional to each other. We could just as easily – and as accurately – refer to $D(\cdot)$ as the wage function.

Lemma 3.3. *For $\alpha \in [0,1]$, (i) When β is sufficiently small, $\frac{\partial D(\alpha,\beta)}{\partial \alpha} < 0$; (ii) When β is sufficiently large, there exists $\tilde{\alpha} \in (0,1)$ such that $\frac{\partial D(\alpha,\beta)}{\partial \alpha} < 0$ for $\alpha < \tilde{\alpha}$ and $\frac{\partial D(\alpha,\beta)}{\partial \alpha} > 0$ for $\alpha > \tilde{\alpha}$. (iii) When β is sufficiently large, $D(\alpha, \beta)$ attains its maximum on $[0,1]$ at $\alpha = 1$.*

Proof: *See Appendix.* □

The aggregate demand function reflects two relationships: the price index and optimal pricing by firms. Part (i) of the Lemma tells us that, when β is small, the aggregate demand function is decreasing in the adjustment probability. To see the intuition, consider the extreme case where $\beta = 0$; in this situation, firms take no account of future periods when setting their current price, so $g(\alpha; 0) = 1$ and $x^*(\alpha)$ is a constant. The adjustment probability then affects the aggregate demand function only through its role as a weight in the price index (see 3.5), and an increase in adjustment probabilities lowers aggregate demand in this case. When $\beta > 0$, however, there is also an indirect effect that acts in the opposite direction. The normalized price charged by adjusting firms is decreasing in the adjustment probability ($\partial g / \partial \bar{\alpha} < 0$); a higher probability of adjustment next period means that firms optimally choose a lower normalized price when they adjust. This fall in $x^*(\bar{\alpha})$ serves to increase aggregate demand. When β is sufficiently large, this effect can more than offset the price index effect, and $D(\alpha, \beta)$ is increasing in α over some of its range. In particular, it decreases up to some $\tilde{\alpha}$ (which is defined in the Appendix), and increases thereafter, attaining its maximum at $\alpha = 1$.[19]

3.4.2. The Best-Response Correspondence

The steady-state best-response correspondence of an individual firm is the correspondence $\alpha(\bar{\alpha})$ implicitly defined by

$$\alpha(\bar{\alpha}) = \arg\max \mathsf{v}(\alpha; s(\bar{\alpha})). \tag{3.30}$$

The steady-state value function $\mathsf{v}(\alpha; s(\bar{\alpha}))$ may possess more than one local maximum. There may be one or more interior local maxima $\alpha^*(\bar{\alpha})$ implicitly defined by

$$\alpha^*(\bar{\alpha}) = \left\{ \alpha : \frac{\partial \mathsf{v}(\alpha; s(\bar{\alpha}))}{\partial \alpha} = 0 \text{ and } \frac{\partial^2 \mathsf{v}(\alpha; s(\bar{\alpha}))}{\partial \alpha^2} < 0 \right\}. \tag{3.31}$$

[19] The behavior of $D(\alpha, \beta)$ for intermediate values of β depends upon the other parameters of the model. Under some circumstances, there are values of β for which aggregate demand is first increasing, then decreasing in β.

We refer to any continuous function $\alpha^*(\bar{\alpha})$ defined by (3.31) as an *interior arm* of the best-response correspondence.[20] In addition, there may be a local maximum at the corner with flexible prices; this occurs when $\partial \mathsf{v}(1; s(\bar{\alpha}))/\partial\alpha > 0$. We refer to this as the *flexible arm* of the best-response correspondence.

The following analysis considers the case where there are two local maxima, one of which is the flexible arm. This is the case that corresponds to our benchmark analysis.[21] The best-response correspondence is given by

$$\alpha = \begin{cases} \alpha^*(\bar{\alpha}) \text{ if } \mathsf{v}(\alpha^*; s(\bar{\alpha})) \geq \mathsf{v}(1; s(\bar{\alpha})) \\ 1 \text{ if } \mathsf{v}(1; s(\bar{\alpha})) \geq \mathsf{v}(\alpha^*; s(\bar{\alpha})) \end{cases}.$$

It is possible for the best-response correspondence to exhibit discontinuities, where, as $\bar{\alpha}$ changes, there is a jump from the interior arm to the flexible arm, or vice versa. As an illustration, consider Figure 1, which considers the case where $\frac{\varepsilon}{\varepsilon-1} = 1.10$, and $\mu = 1.085$. Panel B of Figure 1 shows $\mathsf{v}(\alpha; s(\bar{\alpha}))$ for the individual firm, illustrated as a function of α for three different values of $\bar{\alpha}$. Notice that $\mathsf{v}(\alpha; s(\bar{\alpha}))$ does indeed possess two *local* maxima: at $\alpha = 1$, which corresponds to flexible prices, and at $\alpha \simeq 0.18$, indicating that the firm will keep its price fixed for two periods over 80% of the time. Panel A displays the steady-state best-response correspondence. When the rest of the economy exhibits price stickiness (low values of $\bar{\alpha}$), the individual firm's best response is flexible prices ($\alpha = 1$). When $\bar{\alpha} = 0.94$, however, the best response of an individual firm jumps to one of substantial price stickiness. Thus there is a discontinuity in the best-response correspondence.[22] In this particular case, because the discontinuity involves a downward jump across the 45 degree line, there is no

[20] The value function may also possess a local maximum at zero: $\partial \mathsf{v}(0; s(\bar{\alpha}))/\partial\alpha < 0$. We know, however, that non adjustment for every draw of the fixed cost cannot be an optimal strategy for firms, and so this cannot ever be a global maximum. We thus ignore such local maxima.

[21] More precisely, for the case where the distribution of menu costs is uniform, extensive numerical simulations indicate that there are at most two local maxima, and when there are two maxima, one of them is associated with flexible prices. We have not been able to rule out the possibility of more than two local maxima, however, since this would require a condition on the third derivatives of the profit sum and the cost sum. The results that we obtain for this case could in principle be modified to apply to discontinuities between any two local maxima, although the sufficient conditions would be considerably more complex.

[22] At the discontinuity the firm is indifferent between two strategies, one entailing a low frequency of adjustment, and the other entailing adjustment every period. This may seem counterintuitive – after all, whenever a firm draws a menu cost, it either finds it worthwhile to adjust, or else it does not. At the point of

pure strategy steady-state equilibrium.

3.4.3. Properties of the Best Response Correspondence

Our analysis proceeds in two steps: we first examine the properties of the interior arm (Lemma 3.4), and then consider the possibility and consequences of discontinuities (Proposition 3.5 and corollary 3.6).

Lemma 3.4. *(i) For small β, the interior arm of the best-response correspondence exhibits complementarity everywhere. (ii) As $\beta \to 1$, the interior arm of the best-response correspondence does not exhibit complementarity at any fixed point; (iii) As $\beta \to 1$, the interior arm of the best-response correspondence has a unique fixed point, α^{**}.*

Proof: See Appendix. □

The first part of the lemma tells us that when firms do not care very much about the future (β small), the interior arm exhibits strategic complementarity. This suggests that, for small β, multiple steady states are possible, because the best-response correspondence may intersect the 45 degree line more than once. The intuition for part (i) is as follows. Complementarity in this model operates only through the profit sum, because this is where aggregate adjustment affects the value of an individual firm. In particular, there is strategic complementarity whenever increased aggregate adjustment raises the marginal effect on profits of individual adjustment. We refer to this marginal effect as the marginal profit sum; it equals

$$\frac{\partial \pi_{SUM}\left(\alpha, s\left(\bar{\alpha}\right)\right)}{\partial \alpha} = \frac{\beta \lambda\left(\bar{\alpha}\right)\left[\pi\left(x^*\left(\alpha\right); s\left(\bar{\alpha}\right)\right) - \pi\left(\frac{x^*(\alpha)}{\mu}; s\left(\bar{\alpha}\right)\right)\right]}{\left(1+\beta\left(1-\alpha\right)\right)^2}. \tag{3.32}$$

The *sign* of the marginal profit sum depends on the sign of the bracketed expression, which is the difference between profits from adjusting and not adjusting. That is, the sign of the marginal profit sum depends on whether a firm earns higher profits by adjusting or not adjusting.

Now, we know from Lemma 3.3 that when the discount factor is near zero, increased aggregate price adjustment ($\bar{\alpha}$) unambiguously reduces aggregate demand, because when

discontinuity, however, an adjusting firm is indifferent between two different *prices* that it could set, each of which implies a different *future* probability of adjustment. The value of the firm will of course be the same whichever price it chooses.

more firms adjust, the general price level rises and aggregate demand falls. But, as noted previously, a decrease in aggregate demand implies a proportional decrease in the real wage, and the net effect is to unambiguously increase (normalized) profits. Higher aggregate adjustment thus raises profits for all firms. This effect applies proportionately to both adjusting and non-adjusting firms (the ratio of profits for an adjusting firm to profits for a nonadjuster is independent of aggregate adjustment), and so higher aggregate adjustment implies a proportionate increase in the profit sum. Furthermore, for small β the marginal profit sum is positive (firms care little about the future, and so adjusting firms will set a price close to the static optimum, thus earning higher profits than they would if they did not adjust their price). An increase in aggregate adjustment therefore corresponds to an increase in the fraction of firms for whom it is optimal to adjust. For low β, there is strategic complementarity, and multiple steady states are a possibility.

But the lemma also tells us that as $\beta \to 1$, it is no longer possible for there to be multiple intersections of the interior arm of the best-response correspondence with the 45 degree line, because complementarity is never present at a fixed point. When β is close to 1, we know from Lemma 3.3 that the effect of aggregate price adjustment on aggregate demand $(\partial D(\bar{\alpha}, \beta) / \partial \bar{\alpha})$ is negative for low $\bar{\alpha}$ and positive for high $\bar{\alpha}$. Moreover, it turns out that the marginal profit sum is similarly negative for low α and positive for high α. When β is high and individual adjustment is low at this fixed point, the firm puts high weight on profits in nonadjusting periods, and optimally earns higher profits in those periods than in adjusting periods.

The intuition for part (ii) of the lemma is thus necessarily more complicated than that for part (i). At a fixed point with low $\bar{\alpha}$, an increase in aggregate adjustment decreases aggregate demand, just as in the low-β case. The lower aggregate demand again corresponds to lower real wages and hence a higher *level* of expected profits. But at this fixed point, α is also low and the firm's marginal profit sum is therefore negative – that is, an increase in the firm's adjustment probability *reduces* the firm's discounted sum of profits. It follows that increased adjustment by other firms reduces the marginal benefit of adjustment for the individual firm, so the firm optimally chooses to adjust less: the interior arm slopes down. At a fixed point with high $\bar{\alpha}$, the situation is reversed, but the conclusion is the same. An increase in aggregate adjustment decreases aggregate demand, raises the profit sum, and lowers the marginal profit sum, which is now positive. This decreases the loss associated from not maximizing the profit sum, and the firm chooses a lower adjustment probability.

Figure 2 provides a heuristic guide to this part of the lemma. The figure is drawn for the case where $\beta = 1$, and divides $(\alpha, \bar{\alpha})$ space into four quadrants. The proof of the lemma shows that the slope of the interior arm depends upon two things: whether $\bar{\alpha}$ (the adjustment probability of all other firms) is smaller or larger than a critical value $\tilde{\alpha}$, and whether α (the adjustment probability of the individual firm) is smaller or larger than a critical value $\breve{\alpha}$. Moreover, when $\beta = 1$, these critical values are identical: $\tilde{\alpha} = \breve{\alpha} = \hat{\alpha}$ (where $\hat{\alpha}$ is defined in $(A.5)$ in the Appendix). It turns out that the interior arm can take one of only two possible forms. One possibility is that it involves relatively low adjustment everywhere ($\alpha < \breve{\alpha}$), in which case it is first decreasing, then increasing. This follows from the fact that the marginal profit sum is positive (since $\alpha < \breve{\alpha}$), whereas the derivative of aggregate demand is negative for $\bar{\alpha} < \hat{\alpha}$ and positive for $\bar{\alpha} > \hat{\alpha}$. Alternatively, the interior arm may involve relatively high adjustment everywhere ($\alpha > \breve{\alpha}$), in which case it is first increasing and then decreasing. Both possibilities are shown in Figure 2. The key observation is that because the 45 degree line lies in the northeast and southwest quadrants, in either case the interior arm has negative slope when it intersects the 45 degree line. There is no strategic complementarity, and so there is a unique intersection.

Why do both the sign of the marginal profit sum and the sign of the effect of aggregate adjustment on aggregate demand flip at the same value $\hat{\alpha}$ when β approaches one? The answer is that, for high β, the firm's expected profit function (the profit sum) behaves with respect to individual adjustment very much like the aggregate demand function behaves with respect to aggregate adjustment. The profit sum is proportional to a weighted average of the demand an adjusting firm faces in the period it adjusts and the discounted probability-weighted demand it expects to face while charging the same price one period after adjusting. Specifically,

$$\pi_{SUM}\left(\alpha, s\left(\bar{\alpha}\right)\right) = \left(\frac{1}{\varepsilon}\right)\left(\frac{1}{D(\bar{\alpha},\beta)}\right) \times \tag{3.33}$$

$$D(\bar{\alpha},\beta)^{2-\varepsilon}\left\{\left(\frac{1}{1+\beta\left(1-\alpha\right)}\right)\left[\frac{\varepsilon\chi}{\varepsilon-1}g\left(\alpha;\beta\right)\right]^{1-\varepsilon} + \left(1-\frac{1}{1+\beta\left(1-\alpha\right)}\right)\left[\frac{\varepsilon\chi}{\varepsilon-1}\frac{g\left(\alpha;\beta\right)}{\mu}\right]^{1-\varepsilon}\right\},$$

where the second line is the aforementioned weighted average, and the objects in square brackets are the normalized prices charged in periods of adjustment and nonadjustment.

Aggregate demand has the same flavor as a weighted average of demand in adjusting and nonadjusting periods, except that the former is a function only of aggregate adjustment, whereas the latter is a function of both individual and aggregate adjustment. Indeed, when $\beta = 1$ and $\alpha = \bar{\alpha}$, so that individual and aggregate adjustment correspond, aggregate demand is identical to the weighted average of a firm's demand across periods. To see this, refer back to (3.4) and (3.3). For $\beta = 1$ and $\alpha = \bar{\alpha}$, the individual firm's weights in (3.33) are identical to the aggregate adjustment fractions ω' and $(1 - \omega')$ in (3.4), and the individual firm's normalized prices across periods in (3.33) are identical to the cross-sectional distribution of normalized prices in (3.4). More generally, we can see from (3.28) that when $\beta = 1$, the derivative of the profit sum with respect to α has the same sign as the derivative of $D(\alpha, \beta)$ with respect to α.

The assumption that $\beta = 1$ is of course an approximation, since if β were truly unity, there would be no equilibria with finite utility. But since the critical values $\tilde{\alpha}$ and $\breve{\alpha}$ are continuous in β, we could draw a figure similar to figure 2 for β near unity. For $\beta \simeq 1$, the quadrants would not meet exactly on the 45 degree line, but there would still be only a small segment of the 45 degree line consistent with an upward-sloping interior arm. Moreover, $\hat{\alpha}$ is independent of the menu cost distribution, $F(\cdot)$, but the *position* of the interior arm varies as $F(\cdot)$ changes, because of the presence of the menu cost term C_{SUM} in (3.25). For most menu cost distributions, the interior arm will not cross the 45 degree line in this range. And as $\beta \to 1$, the segment vanishes, so the interior arm will not lie in this range for almost all possible menu cost distributions.

Figure 3 illustrates two best-response correspondences drawn for different values of β.[23] In one case, β is very small ($\beta = 0.1$), the best-response function is increasing, and there are multiple steady states. The other best-response function in Figure 3 is drawn for $\beta = 0.5$, and shows that the optimal adjustment policy of an individual firm is almost invariant to the adjustment patterns of other firms.

The lemma thus suggests that we are unlikely to observe multiple steady state equilibria for high values of the discount factor. Specifically, the best response correspondence is one dimensional, and therefore if it is continuous, a necessary condition for there to be multiple

[23] The parameters used to generate Figure 3 are: $\varepsilon = 21$, $\mu = 1.042$, and $\chi = 3$. The menu cost distribution is uniform with $B = 0.01$.

steady state equilibria is that this function have slope greater than unity at a fixed point (see Cooper and John [1988]).

The lemma does not prove that there are no multiple equilibria, however, because the best-response correspondence may be discontinuous. Discontinuities mean that multiple steady states might still be possible if there is a jump from the interior arm to the flexible arm. Discontinuities also mean that it is possible that there is no steady state equilibrium (at least in pure strategies).

Let β be sufficiently large that the interior arm has a unique fixed point (see Lemma 3.4). Denote this fixed point by α^{**}. As $\beta \to 1$, two conditions can be used to summarize the set of steady state equilibria. They are

$$\mathsf{v}\left(\alpha^{**}, s\left(\alpha^{**}\right)\right) > \mathsf{v}\left(1, s\left(\alpha^{**}\right)\right) \tag{3.34}$$

$$\mathsf{v}\left(1, s\left(1\right)\right) > \mathsf{v}\left(\alpha^{*}\left(1\right), s\left(1\right)\right) \tag{3.35}$$

If (3.34) holds, the fixed point of the interior arm is a steady state equilibrium, and if (3.35) holds, flexible prices is a steady state equilibrium. If both conditions hold, there are two steady state equilibria, and if neither condition holds, there is nonexistence of symmetric steady state equilibrium.

The following proposition sets out necessary conditions for multiple equilibria that arise from discontinuities of the best-response correspondence.

Proposition 3.5. *Let β be sufficiently large that the interior arm has a unique fixed point (see Lemma 3.4). Denote this fixed point by α^{**}. Let $\hat{\alpha}$ be as defined in the Appendix (A.5). As $\beta \to 1$, necessary conditions for multiple equilibria are:*

(i) $\alpha^{**} < \hat{\alpha}$

(ii) $\mathsf{v}\left(\alpha^{*}\left(\hat{\alpha}\right), s\left(\hat{\alpha}\right)\right) < \mathsf{v}\left(1, s\left(\hat{\alpha}\right)\right).$

Proof: See Appendix. □

The intuition for the proposition is as follows. For $\bar{\alpha} < \hat{\alpha}$, any discontinuities take the form of "upward" jumps – from the interior arm to the flexible arm as $\bar{\alpha}$ increases. For $\bar{\alpha} > \hat{\alpha}$, the converse is true: any discontinuities are "downward" jumps from the flexible arm to the interior arm. Multiple equilibria require upward jumps, and so multiple equilibria are possible only if the best-response correspondence initially follows the interior arm, crosses the 45 degree line, and then jumps up to the flexible arm before $\bar{\alpha} = \hat{\alpha}$. Hence multiplicity is only possible if there is an interior equilibrium at some $\alpha^{**} < \hat{\alpha}$, and then a discontinuity in the range $\bar{\alpha} \in (\alpha^{**}, \hat{\alpha}]$. Conditions (i) and (ii) are necessary and sufficient for this (even these are not sufficient for multiplicity, however, for there could be a downward jump above $\hat{\alpha}$). A violation of (i) or (ii) therefore rules out multiple equilibria.

Checking (i) and (ii) requires solving for the interior fixed point, which may not be completely straightforward. We can however identify some simpler conditions for ruling out multiplicity.

Corollary 3.6. *As $\beta \to 1$, multiple symmetric steady state equilibria are ruled out if*

(a)

$$\left(\frac{1}{\varepsilon}\right)\left[\left(\frac{1+\mu^{\varepsilon}}{1+\mu^{\varepsilon-1}}\right)^{\varepsilon}\left(\frac{2}{1+\mu^{\varepsilon}}\right)-1\right] > \chi E\left(\xi\right).$$

or

(b)

$$\chi E\left(\xi\right) - C_{SUM}(\hat{\alpha}) > \left(\frac{1}{\varepsilon}\right)\left[\frac{\left(\mu^{\varepsilon}-1\right)}{\varepsilon\left(\mu-1\right)\mu^{\varepsilon-1}} \cdot \left(\frac{\left(\mu^{\varepsilon}-1\right)\left(\varepsilon-1\right)}{\left(\mu^{\varepsilon-1}-1\right)\varepsilon}\right)^{\varepsilon-1} - 1\right]$$

Proof: *See Appendix.* $\square$

Condition (a) rules out multiplicity for a substantial region of the parameter space. For example, if we assume (as in our simulations below) that $E\left(\xi\right) = 0.005$, then the condition holds unless inflation exceeds about 17% per period. As the markup increases, it is still true that the condition fails to hold above $\mu \simeq 0.17$, and it also ceases to hold at low levels of inflation. Thus, for a markup of 10%, the condition holds for inflation between about 7 and 17%. When the markup reaches about 44%, the condition no longer holds. Condition (b) depends upon the distribution of menu costs.

3.5. The Number of Steady-State Equilibria: A Numerical Approach

Proposition 3.5 establishes that, under certain conditions, multiple steady states are unlikely to occur for values of the discount factor close to 1. The proposition naturally leads one to speculate that multiple steady states will not arise for a wide range of values for the other parameters of the model. Here we provide information on how the number of steady-state equilibria varies with the demand elasticity (ε) and the inflation rate (μ) for our benchmark case, and below we present sensitivity analysis.

3.5.1. Computation and Calibration

For given parameter values, $c(\bar{\alpha})$, $w(\bar{\alpha})$ and $\lambda(\bar{\alpha})$ are given by (3.29), together with $\lambda(\bar{\alpha}) = 1/c(\bar{\alpha})$ and $w(\bar{\alpha}) = \chi c(\bar{\alpha})$. Profits and a firm's value as functions of $\bar{\alpha}$ follow from (3.23) and (3.25). Using a fine grid for $\bar{\alpha}$, we compute the fixed point(s) of the best response correspondence (3.30). Regarding Assumption 1, we choose parameters so that $J = 2$ for most of our parameter space, checking to see that this condition is in fact satisfied. For a subset of the cases where the condition is not satisfied, we check for equilibria with $J = 3$, and present results on the number of equilibria.

The distribution function for fixed costs of price adjustment is not something for which there are sharp estimates available: we assume that ξ is drawn from a beta distribution on $(0, B)$, and experiment with the mean and variance (equivalently, the exponents commonly referred to as ALPHA and BETA in the beta c.d.f.), while keeping B at 0.01, a value that is small relative to the labor input used in producing final goods. The beta distribution is flexible, and allows us to examine the robustness of our results to different assumptions about the costs of changing prices. Throughout, we fix the preference parameter β at 0.975, which makes it natural to interpret one period in the model as six months. For fixed values of β, $F(\xi)$, ψ, and χ, we find the number of symmetric steady-state equilibria at each point in a 30 x 30 grid of $(\frac{\varepsilon}{\varepsilon-1} = 1.01, 1.02, ..., 1.30)$ and $(\mu = 1.005, 1.01, ..., 1.15)$.[24] Our grid thus covers specifications of market power and inflation levels that are reasonable for developed low-inflation economies.

3.5.2. The Benchmark Case

Figure 4 displays the number of steady-state equilibria for our benchmark case. For this figure, we assume that the menu cost distribution is uniform (this is a special case of the beta distribution with ALPHA$= 1$ and BETA$= 1$). The various characters in the grid should be interpreted as follows. Nonnegative integers indicate the number of steady-state equilibria in which no price is ever fixed for more than two periods; an "F" (for Flexible prices) means that there is a unique symmetric, pure strategy steady-state equilibrium (SPSSE) with $\alpha = 1$,

[24] We emphasize $\frac{\varepsilon}{\varepsilon-1}$ instead of ε because $\frac{\varepsilon}{\varepsilon-1}$ is the average markup when there is no price stickiness, and hence is easier to interpret than the demand elasticity.

and a "–" means that some firms with one-period old prices would choose to keep their prices fixed for another period.

Certain aspects of Figure 4 are easy to interpret. First, at higher rates of inflation, prices become less sticky. This implication of state-dependent pricing models has previously been discussed by Ball, Mankiw and Romer [1988] and Dotsey, King and Wolman [1999]. (Similarly, at very low rates of inflation – the "—" region – firms would want to keep their prices fixed for more than two periods.) Second, price stickiness tends to increase with market power. At low values of $\frac{\varepsilon}{\varepsilon-1}$, which correspond to high values of ε, a firm whose price is significantly below the price level would be swamped with demand, and would have to meet that high demand at a suboptimal price. Such a firm would be willing to incur a relatively high menu cost to adjust its price. Equivalently, a firm would not wish to set a high price, planning to leave it in place for two periods because it would see its sales fall substantially in the first period.

In the "-" region of Figure 4, in order to know how many steady state equilibria exist, it is necessary to consider candidate steady state equilibria with some prices fixed for more than two periods. It is feasible to generalize the approach used above, and to compute two-dimensional best response correspondences. These yield α_1 and α_2, the optimal adjustment probabilities for firms that enter the period with prices set one and two periods ago, respectively, as functions of $\bar{\alpha}_1$ and $\bar{\alpha}_2$, the corresponding aggregates. For a 16-element grid in the "-" region, we have computed the fixed points of this best response correspondence. The grid consists of $\varepsilon/(\varepsilon-1) = \{1.01, 1.11, 1.20, 1.30\}$ and $\mu = \{1.01, 1.023, 1.037, 1.05\}$, and at every point on this grid there is a unique steady state equilibrium. In this region, as in our earlier analysis, we find that individual adjustment decisions are relatively unresponsive to aggregate adjustment decisions; we find that the two-dimensional best-response correspondence is still "flat."

The one feature of Figure 4 which is not self-explanatory is the significant number of zeros present, indicating nonexistence of symmetric steady state equilibrium with pure strategies. The zeros in the far left of the figure – where inflation is high and markups are low – arise simply because firms have little market power, earn low profits, and so cannot cover the fixed costs of changing their price frequently. The zeros that separate sticky prices from flexible prices reflect discontinuities in the best-response correspondence, as illustrated in figure 1.

These discontinuities take the form of downward jumps across the 45 degree line. Although it is not clear from Figure 4, the range of nonexistence between sticky prices and flexible prices is not just a knife edge case.

As described earlier, a discontinuity in the steady state best-response correspondence indicates that an adjusting firm is indifferent between two pricing policies. One involves setting a relatively low price and adjusting again with certainty in the next period. This policy yields high profits but entails high adjustment costs. The other policy involves setting a relatively low price and only adjusting again in the next period if the adjustment cost is low enough; this yields low profits but also entails low adjustment costs. As the aggregate adjustment probability $\bar{\alpha}$ is imagined to increase past the point of discontinuity, the value-maximizing adjustment policy for an individual firm flips from completely flexible prices to a policy of adjusting only for a low enough draw of the adjustment cost. Seen in this way, it is clear that nonexistence represents more than a knife-edge set of cases. If there is nonexistence associated with a discontinuity for one set of parameters, a marginal change in one of the parameters will generally yield another case of nonexistence associated with a discontinuity.

The nonexistence associated with these discontinuities is only nonexistence of steady state equilibrium with *pure strategies*; there is almost certainly a steady state equilibrium near the point of discontinuity, in which firms randomize according to a particular distribution, and are indifferent between two policies whenever they adjust. However, standard approaches in the business cycle literature are to linearize around a deterministic steady state, or to use a deterministic steady state as a starting point for computing an equilibrium with uncertainty. These approaches would have to be modified if one were interested in studying the model's behavior in the zeros region on the border between flexible and sticky prices. Furthermore, a naive approach to computing steady state equilibrium would miss the nonexistence, and could even conclude that there were *two* steady state equilibria. That naive approach – alluded to in section 3.2.2 above – avoids explicitly considering a firm's value maximization; it instead uses the condition which balances the cutoff cost below which a firm with a one-period old price adjusts, against the value of adjusting when entering the period with a one-period old price:

$$\begin{aligned}\chi F^{-1}(\alpha) &= \lambda(\bar{\alpha})\cdot\left(\pi\left(x^*(\alpha);\bar{\alpha}\right)-\pi\left(x^*(\alpha)/\mu;\bar{\alpha}\right)\right)\\ &\quad -\beta(1-\alpha)\,\mathsf{v}\left(\alpha;s(\bar{\alpha})\right)\\ &\quad +\beta\chi\left(E\xi-E\left(\xi|\xi<F^{-1}(\alpha)\right)\right). \end{aligned} \tag{3.36}$$

This is the steady state version of (2.20), where we have used (3.24) to simplify. In the region of nonexistence in figure 4, there are two solutions ($\alpha = \bar{\alpha}$) to (3.36), but neither of them represent equilibria. Referring to Figure 1.B, these two solutions are indicated by points on the steady state value function with a zero derivative.

3.6. Sensitivity Analysis

We now report some numerical sensitivity analysis. We vary the distribution of fixed costs, the labor supply elasticity and the money demand specification, and we modify the consumption aggregator function (2.1) so that it does not imply constant demand elasticities.

Figure 5 sets the parameters of the beta distribution to ALPHA= BETA= 10. This maintains symmetry, and makes the distribution of fixed costs s-shaped (the p.d.f. bell-shaped), with low variance. We might expect an s-shaped distribution to generate multiple SPSSE, because over a certain range, an s-shaped distribution has the property that $F'()$ is very high. Intuitively, a steep distribution function might lead to complementarity: a firm's adjustment probability is equal to the probability that it draws an adjustment cost less than the gain from adjusting. If the cdf is steep, then the firm's optimal adjustment probability is sensitive to the value of adjusting. In fact, Figure 5 shows that this modification has little effect, and we still do not observe any cases of multiple SPSSE.

We have experimented with other parameter values for the beta distribution and obtained similar results. To help explain these findings, Figure 6 displays best-response functions that correspond to a point in the middle of Figure 4 and Figure 5. In both cases the best-response function is almost completely flat. In the uniform case, however, it is shifted up so that the steady state has $\alpha_1 = \bar{\alpha}_1 \approx 0.26$ as compared to $\alpha_1 = \bar{\alpha}_1 \approx 0.005$ when the distribution is concentrated. In equilibrium, the benefit to adjusting price is in large part pinned down by factors other than the distribution of fixed costs. When the mass of the distribution is heavily concentrated about the mean, a smaller fraction of firms will draw costs that make it worthwhile to adjust. Figure 6 thus reveals that while modifying the distribution of fixed costs does not change the *number* of equilibria, it does affect the degree of price stickiness.

Figure 7 deviates from Figure 4 by changing the preference specification so that the labor supply elasticity is unity rather than infinity. As with changing the distribution, changing the labor supply elasticity does not significantly alter the pattern of SPSSE, although it does lead to somewhat greater price rigidity.

Figure 8 replicates Figure 4, except that money demand is explicitly motivated by a cash-in-advance constraint. Again, there is little difference between the two figures. Minor differences are attributable to the fact that in a cash in advance framework, the steady-state ratio of consumption to the real wage is given by $(\chi \cdot (1+R))^{-1}$, where R is the nominal interest rate. Higher inflation corresponds to a higher nominal rate, which decreases the gain from adjusting. The region of price stickiness in Figure 8 thus extends to higher levels of inflation than when money demand is modeled in an *ad hoc* manner.

Finally, we modify the consumption aggregator function from (2.1) to

$$c_t = \left[\int_0^1 (c_t(z) + \overline{c})^{\frac{\varepsilon-1}{\varepsilon}}\, dz\right]^{\frac{\varepsilon}{\varepsilon-1}} - \overline{c}. \tag{3.37}$$

With this aggregator, the elasticity of demand for good z is increasing in the good's relative price, instead of being constant. The demand for good z is

$$c\,(z) = \left(\frac{P\,(z)}{P}\right)^{-\varepsilon} (c + \overline{c}) - \overline{c},$$

the demand elasticity is

$$\epsilon = \left|\frac{\partial \ln c\,(z)}{\partial \ln\left(\frac{P(z)}{P}\right)}\right| = \varepsilon \cdot \left(1 - \frac{\overline{c}}{\left(\frac{P(z)}{P}\right)^{-\varepsilon}(c+\overline{c})}\right)^{-1},$$

and at a relative price of unity, the elasticity of the demand elasticity is

$$\left.\frac{\partial \ln \epsilon}{\partial \ln\left(\frac{P(z)}{P}\right)}\right|_{\frac{P(z)}{P}=1} = -\varepsilon \cdot \frac{\overline{c}}{c}.$$

For the constant elasticity aggregator (2.1), computations are made easy by the fact that for given aggregate adjustment probabilities, all other aggregate variables have closed form solutions. Closed form solutions are not available for the modified aggregator (3.37) – essentially, one cannot solve explicitly for the optimal normalized price as in (3.18). However, it is still possible to compute the steady state levels of demand and the real wage corresponding to any aggregate adjustment pattern, and then determine optimal behavior by a firm confronted with these aggregates. Figure 9 shows four examples of the best-response correspondence for different points in markup-inflation space. In each case, we illustrate three different values of $\bar{c}$. While different values of this parameter do affect the position of the best-response correspondence – in particular, increases in $\bar{c}$ tend to lead to greater flexibility of prices – we find, as before, that the individual adjustment decision is insensitive to the choices of other firms. Again we find examples of non-existence.

In sum, our numerical analysis uncovers no evidence of multiple steady states for standard calibrations of the model with plausible values for the discount factor. Multiplicities occur only for very low values of the discount factor that are not relevant for business cycle analysis. These findings are consistent with our analytical work, which suggested that multiple equilibria were unlikely to be observed for β close to 1. On the other hand, nonexistence of pure-strategy equilibrium arises in a small region of the parameter space.

3.7. Yeoman-Farmers

This section considers a different form of sensitivity analysis: we ask if the nature of the labor market is critical for our results. We replace the competitive labor market with "yeoman-farmers" (as in the Ball-Romer analysis) who produce output using their own labor. Our yeoman-farmer model is identical in all respects to our basic set-up, except that each firm is owned by an individual agent who produces output using her own labor. We assume slightly more general preferences than in our benchmark analysis by allowing for a non-infinite elasticity of labor supply:[25] $u(c, n) = \ln c - \chi n^{\nu}$. Agent $k's$ aggregate consumption simply equals her total income, which in turn simply equals her revenue from the sales of her output: $c_k = p_k y_k$. The agent faces the demand curve $y_k = (p_k)^{-\varepsilon} y$, where y is total income in the economy. We suppose that the aggregate state is constant, and, as before, we let $x_t^k \equiv P_t^k / M_t$. Finally, $M_t = P_t y_t$.

[25] This version of the yeoman farmer model is not identical to that in Ball and Romer's original paper, because they assumed that preferences were linear in consumption.

The solution of the model is in the Appendix. There, we show that the steady-state value function is

$$\begin{aligned} \mathsf{v}^{y.f.}\left(\alpha, s\left(\bar{\alpha}\right)\right) &= \left(\frac{1}{\varepsilon}\right)\ln y\left(\bar{\alpha}\right) + \frac{1-\varepsilon}{\varepsilon\nu}\left[\ln\left(\frac{\varepsilon\nu\chi}{\varepsilon-1}\right) + \ln q(\alpha)\right] + \frac{\beta\left(1-\alpha\right)\left(\varepsilon-1\right)\ln\mu}{\left(1+\beta\left(1-\alpha\right)\right)} \\ &\quad -\frac{\varepsilon-1}{\varepsilon\nu} - \frac{\beta\chi\left[\alpha E\left(\xi|\xi<F^{-1}\left(\alpha\right)\right)+\beta\left(1-\alpha\right)E\xi\right]}{\left(1+\beta\left(1-\alpha\right)\right)}. \end{aligned}$$

Our key result is immediate from this expression. The aggregate adjustment by other farmers, $\bar{\alpha}$, enters in one place only, and in a separable way. Thus $\frac{\partial^2 \mathsf{v}^{y.f.}(\alpha, s(\bar{\alpha}))}{\partial\alpha\partial\bar{\alpha}} = 0$, and so an individual farmer's choice of adjustment probability is independent of the actions of other farmers. The best-response correspondence is completely flat, and so there must be a unique steady state in this version of the model.

In the yeoman-farmer model, one of the general equilibrium linkages is broken because there is no labor market. The linkage through aggregate demand remains. It turns out, however, that the optimal choice of output in this model is independent of the level of aggregate demand: any increase in demand is offset completely by an increase in the optimal price. An increase in aggregate demand also translates into an increase in income for any individual farmer, but because consumption is logarithmic, this simply shifts the steady-state value function uniformly upwards and hence has no effect on the farmer's adjustment decision.

This result confirms that our basic result is not driven by the assumptions we have made about the labor market. For other specifications of preferences, it might be the case that multiplicity would still arise in the yeoman-farmer model; we have not investigated this model with general preferences. Our result does allow us to assert, however, that an assumption of yeoman-farmers is insufficient to generate complementarity or multiplicity.

4. A One-time shock

We motivated this paper in part with reference to Ball and Romer [1991], who argued that fixed costs of price adjustment lead to multiplicity of equilibria. In this section we study the equilibrium response to an unanticipated monetary shock, which is the experiment analyzed by Ball and Romer. In order to increase comparability to Ball and Romer, we perform this analysis initially in a finite horizon model: if the shock occurs in the final period then we have the static approach taken by Ball and Romer. Backward induction can be used to solve the model for successively longer horizons, and this process converges to yield the equilibrium of our basic infinite horizon model. By basing our solution method on finding all the fixed points of firms' best response functions, in principle we are able to uncover multiple equilibria even in the infinite horizon case.

The discussion here pertains to a one-time shock, followed by constant money growth. We have also computed equilibrium for a case where money growth follows a two-state Markov process, and there is no qualitative difference in the results. The algorithm of course is modified to account for the stochastic money growth, but the modifications are straightforward in the case of a two-state Markov process.

4.1. A Shock in the Final Period (static case)

Ball and Romer showed that in a static model, there was always complementarity in price setting in response to a monetary shock. The one-period version of our model differs from Ball and Romer in that there is an economywide labor market here, and preferences take a different form. Nonetheless, complementarity is present in the final period under the condition that

$$x_T^* < \frac{\varepsilon\chi}{\varepsilon - 1}\mu_T, \tag{4.1}$$

where μ_T is the money growth rate in the final period, and x_T^* is the predetermined nominal price, normalized by the previous period's money supply. That is, if the monetary shock is sufficiently large – relative to the preset price – then the final-period probability that an individual firm chooses to adjust its price is increasing in the probability that the representative firm is adjusting its price. A key to deriving this result is that a firm's optimal price in the final period is independent of the prices set by other firms, and independent of the number of firms adjusting; it is given by

$$x_{T+1}^* = \eta\chi, \text{ where } \eta \equiv \frac{\varepsilon}{\varepsilon - 1}. \tag{4.2}$$

Together, these two expressions imply that if the pre-set price is below the final period optimal price, then there is complementarity.

The formal derivation of (4.1) is in appendix C; here we offer a guide to that derivation and then a more intuitive explanation. The monopolistic competition framework together with the economywide labor market makes the optimal nominal price a fixed markup over nominal marginal cost. With the infinite labor supply elasticity preferences in (3.21), nominal marginal cost is proportional to nominal demand, so that the optimal nominal price is a fixed markup over nominal demand (i.e., a fixed markup over the money supply). An increase in the fraction of firms adjusting (to their optimal price) then has only a direct effect on the price level, raising it. With a given level of the money supply, a higher price level decreases aggregate demand. The direct effect of lower aggregate demand is to decrease the firm's profits – whether or not it adjusts. However, as discussed in our steady-state analysis, the lower aggregate demand also implies lower real wages in general equilibrium, and, for profits denominated in utility terms, this effect dominates. Meanwhile, if we evaluate costs of adjustment in utility terms, they are independent of the adjustment decisions of other firms. To summarize, we have at this point that an increase in the adjustment probability of all firms raises the profits of an individual firm – whether or not it adjusts. Because the effect of aggregate adjustment on profits is proportional to the level of profits, an increase in aggregate adjustment raises the reward to adjusting, and we have the result that there is complementarity in price adjustment as long as the optimal form of adjustment is an increase in the firm's price (note the similarity to the steady state analysis with $\beta = 0$).

More intuitively, the asymmetric nature of complementarity occurs because the profit function is asymmetric in the case of constant elasticity demand functions and constant marginal cost. Profits decrease more steeply for prices that are too high than for prices that are too low. Put another way, convexity of marginal revenue (in terms of price) coupled with constant marginal cost means that the penalty for a price that is too low exceeds the penalty for a price that is correspondingly high.[26] When the money supply rises, a firm's optimal price rises. Greater increases in the aggregate price level make the firm's price too low, and this is more costly than keeping a price that is too high when other firms lower their price in response to a decrease in the money supply. Devereux and Siu (2003) contains a related discussion.

It is helpful to consider why the complementarity we just described is not typically present in our steady state analysis. Unlike in the final period (static case), in steady state an increase in the fraction of firms adjusting has both a direct and an indirect effect on the

[26] Contrast this to the case of linear demand, in which case the profit function is symmetric.

price level. The direct effect, as in the static case, is to increase the price level by putting more weight on the relatively high price set by adjusting firms. The indirect effect, not present in the static case or in steady states with $\beta = 0$, is to *decrease* the price level by inducing adjusting firms to set a lower price (the indirect effect corresponds to $\partial g / \partial \alpha < 0$). This indirect effect tends to work against complementarity.

Although complementarity in the final period is an analytical result, it is not possible to solve analytically for the two functions describing final period equilibrium. The pricing function $x_{T+1}(x_T, \omega_T)$ is the trivial function given in (4.2), but the adjustment function $\omega_{T+1}(x_T, \omega_T)$ must be computed numerically. Our approach is as follows. We specify a finite grid for the state variables ω_T and x_T, and for each point on the grid compute the fixed points of (C.9); the fixed points are the equilibrium values of α_T, and then $\omega_{T+1}(x_T, \omega_T)$ can be recovered from (C.7). If there are multiple fixed points for some region of the state space, then there are multiple equilibria in that region, and the model implies nothing about which equilibrium will occur. To use backward induction to solve the model for a longer horizon, we must make an assumption about equilibrium selection in cases of multiple equilibrium. We assume that in the event of multiple equilibria in a range of the state space in any period t, there is an exogenous distribution over the equilibria, and that distribution is common knowledge. Solving the model for a longer horizon also requires that we compute the value function for an individual firm at each point in the state space:

$$v_T(x, \xi; s_T) = \max \left\{ \begin{array}{c} [c_T(s_T)]^{1-\varepsilon} \cdot (\eta\chi)^{-\varepsilon} [\eta\chi - \chi] - \chi \cdot \xi, \\ [c_T(s_T)]^{1-\varepsilon} (x/\mu_T)^{-\varepsilon} [x/\mu_T - \chi] \end{array} \right\}. \tag{4.3}$$

These value functions are necessary for determining optimal firm behavior in the previous period.

4.2. Using Backward Induction to Solve the Model with an Arbitrary Horizon

The backward induction approach we take assumes that money growth is constant in all future periods. Because we compute equilibrium for arbitrary initial conditions however, we naturally compute the equilibrium response – including transitional dynamics – to an unexpected one-period shock to the money growth rate. Suppose the equilibrium functions $x_{T+1-j}(x_{T-j}, \omega_{T-j})$ and $\omega_{T+1-j}(x_{T-j}, \omega_{T-j})$ and the value functions $v_{T-j}(x, \xi; s_{T-j})$

are known for some $j \geq 0$, given that period T is the final period. We can compute the equilibrium functions $x_{T-j}\left(x_{T-j-1}, \omega_{T-j-1}\right)$ and $\omega_{T-j}\left(x_{T-j-1}, \omega_{T-j-1}\right)$, and the value functions $v_{T-j-1}\left(x, \xi; s_{T-j-1}\right)$ using a generalization of the strategy pursued in the final period. A summary of the computational algorithm is as follows:

For each point on the grid of state variables:[27]

1. Loop over a grid of candidate ω_{T+1-j}, call them ω^i_{T+1-j}

2. For each ω^i_{T+1-j}, loop over a grid of candidate x_{T+1-j}, call them x^k. Compute the pricing best response function for an adjusting firm,

$$x^{b,k}_{T+1-j}\left(x^k_{T+1-j}; \omega^i_{T+1-j}, s_{T-j-1}\right). \tag{4.4}$$

   This computation involves using the firm's value function in the subsequent period to evaluate the future return associated with each possible price chosen in the current period. Fixed points of this response function now form refined candidate prices, corresponding to the candidate adjustment fractions. Thus we have a candidate $d+1-$tuple, $\left\{\omega^i_{T+1-j}, x^i_{T+1-j}\right\}$, where d is the number of fixed points of the pricing best response function (4.4), and x^i_{T+1-j} is a $d-$dimensional vector.

3. For each candidate $d+1-$tuple, $\left\{\omega^i_{T+1-j}, x^i_{T+1-j}\right\}$, compute the firm's optimal adjustment probability. The firm's optimal adjustment probability can be used to calculate a best response function for the fraction of firms $\hat{\omega}_{T+1-j}$ that would adjust if they pursued the policy that is optimal for the individual firm. Fixed points of this response function are the equilibrium values of ω_{T+1-j} for this particular point in the state space.

To reiterate, if either of the best response functions have multiple fixed points, then an exogenous set of probabilities is assumed to determine which fixed point will occur, and

[27] This algorithm can be made arbitrarily accurate by increasing the fineness of the grids. However, computational requirements quickly become unmanageable; for a grid of 65 by 65 points, it takes approximately seven hours to compute the equilibrium for one period, using GAUSS 3.5 on a pentium III 800 mhz dual processor.

those probabilities are common knowledge. Our benchmark assumption is that each fixed point is equally likely.

The analytical finding of complementarity in the final period suggests that there may be multiple equilibria in the one-period model. We have in fact found such multiplicity (which is the form of multiplicity described by Ball and Romer) but it seems to be confined to a small region of the state space. As the horizon lengthens, the incidence of multiplicity has decreased in our example calculations, and we have found no examples of multiple equilibrium responses to a shock in the infinite horizon model.[28] Figure 10 illustrates how the best response function $\widehat{\alpha}_{T+1-j}\left(\alpha_{T+1-j}\right)$ evolves with j at particular points in the state space, for four examples from our benchmark parameterization.[29] Each of the curves in this figure represents a best response function in the impact period of the shock; the curves within a panel differ according to the model's horizon. For example, the curves containing circles represents a static model (one-period), and the curves with plus signs represent an infinite horizon model. While we find no examples of multiple equilibrium responses to a shock in the infinite horizon model, neither do we find that in all cases complementarity vanishes in the infinite horizon model. This is consistent with our steady-state setting, where we found weak complementarity over some range.

Interpreting the best response functions for a shock in the infinite horizon model is less straightforward than interpreting either their steady state or static analogues. In contrast to the steady state case, more aggregate adjustment does not necessarily correspond to a lower price chosen by adjusting firms. But, in contrast to the static case, different levels of aggregate adjustment generally do correspond to different prices chosen by adjusting firms. Referring back to our description of computation, one can see the difficulty involved in interpreting these response functions: each point on one of the best response functions for α in Figure 10 represents a fixed point of a best response function for the normalized price (x). Thus, the slope of the best response function for α reflects the effect of the aggregate

[28] Using backward induction, the equilibrium functions converge to those of the infinite horizon model after six or seven periods. The firm's value functions converge much more slowly, given the discount factor of 0.975, but convergence of value functions is not necessary for convergence of equilibrium aggregate outcomes.

[29] Our computational algorithm, described above, emphasizes ω, the adjustment fraction, rather than α, the conditional adjustment probability. The figure instead illustrates α, for comparability with our earlier discussion of steady state equilibria.

adjustment probability on the location of the fixed point of the best response function for x.

Because of the lack of analytical results, and the small number of examples we have calculated, our analysis of the equilibrium response to a monetary shock is necessarily less conclusive than that involving steady states. While we are not optimistic about the prospects for analytical progress, as computational costs fall it will be feasible to study an ever wider range of examples.

5. Conclusions

Models with fixed costs of price adjustment represent an attractive framework for studying business cycles and the effects of monetary policy. Simple versions of such models are now available for these purposes. However, little is known about the existence and uniqueness of equilibrium in these discrete-time models. Furthermore, Ball and Romer [1991] argued strongly that models with fixed costs of price adjustment are characterized by complementarity and hence can contain multiple equilibria. Their analysis was nonetheless limited by being essentially static. We have extended Ball and Romer's analysis to an explicitly dynamic framework that fits directly into current business cycle research. By so doing, we have considered whether or not complementarities and multiplicities are likely to arise in the context of modern business cycle models.

The model that we study has the potential for multiple steady states and multiple equilibria in response to a shock. We study each of these possibilities, and find that Ball and Romer's conjecture does extend to a dynamic setting, but only in a very limited sense. When the discount factor is very small, we do find complementarity, and there may indeed be multiple steady states.[30] But for large (business cycle) values of the discount factor, the model lacks strong enough complementarity in price adjustment to generate multiple steady states. (In related work, we have found that a similar lack of complementary leads to the finding that cyclical equilibria (e.g. synchronization) do not typically arise when the fundamentals are constant.) In response to a one-time shock – the experiment analyzed by Ball and Romer [1991] – in an infinite horizon model it is no longer the case that complementarity in

[30] This may not seem that surprising, because a small discount factor means that the future matters little to firms, so their behavior is not that different from the static case. As against that, there are steady-state restrictions in the dynamic model that are not present in a static model, even with a low discount factor.

price-adjustment is unambiguously present. Complementarity can arise, but we have found no examples of multiple equilibrium responses to a shock in the infinite horizon model.

Our results suggest that research using state-dependent pricing models is unlikely to be hindered by the presence of multiplicity. Non-existence of pure-strategy equilibrium is possible in a small region of the parameter space, however, and researchers should take care to use solution methods that will not stumble on such non-existence. Based on our findings, it also does not seem that – at least in this class of models – multiplicity will be central to explaining different degrees of price stickiness experienced by economies with apparently similar fundamentals. On the other hand, the work by Caplin and Leahy [1991] and Caballero and Engel [1993] highlights the fact that differing responses to monetary shocks across time and locations can be consistent with a unique equilibrium under state-dependent pricing, as the response to shocks depends on the state of the economy.[31] Future research may be fruitfully directed at quantitative analysis using state-dependent pricing models to analyze these issues further.

[31] Burstein [2002] documents the state-dependent nature of the response to shocks in a richer version of the model in this paper.

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A. Proofs of propositions and lemmata for steady state

A.1. Preliminaries

1. We will consider many functions of the form

$$\rho\left(\alpha,\beta;\mu,a,b\right)=\frac{1+\beta(1-\alpha)\mu^{a}}{1+\beta(1-\alpha)\mu^{b}} \tag{A.1}$$

where a and b are parameters. Note that for $a>b$,

$$\frac{\frac{\partial\rho(\alpha,\beta;\mu,a,b)}{\partial\alpha}}{\rho\left(\alpha,\beta,\mu;a,b\right)}=\frac{-\beta\left(\mu^{a}-\mu^{b}\right)}{\left(1+\beta(1-\alpha)\mu^{a}\right)\left(1+\beta(1-\alpha)\mu^{b}\right)}<0 \tag{A.2}$$

2. Now, $r(\alpha,\beta)=\rho\left(\alpha,\beta;\mu,\varepsilon-1,0\right)$, so

$$\frac{\frac{\partial r(\alpha,\beta)}{\partial\alpha}}{r(\alpha,\beta)}=\frac{-\beta\left(\mu^{\varepsilon-1}-1\right)}{\left(1+\beta\left(1-\alpha\right)\mu^{\varepsilon-1}\right)\left(1+\beta\left(1-\alpha\right)\right)}<0, \tag{A.3}$$

and

$$\frac{\frac{\partial r(\alpha,1)}{\partial\alpha}}{r(\alpha,1)}=\frac{-\left(\mu^{\varepsilon-1}-1\right)}{\left(1+\left(1-\alpha\right)\mu^{\varepsilon-1}\right)\left(2-\alpha\right)}<0.$$

3. In addition, $g(\alpha,\beta)=\rho\left(\alpha,\beta;\mu,\varepsilon,\varepsilon-1\right)$, so

$$\frac{\frac{\partial g(\alpha,\beta)}{\partial\alpha}}{g(\alpha,\beta)}=\frac{-\beta\mu^{\varepsilon-1}\left(\mu-1\right)}{\left(1+\beta\left(1-\alpha\right)\mu^{\varepsilon-1}\right)\left(1+\beta\left(1-\alpha\right)\mu^{\varepsilon}\right)}<0. \tag{A.4}$$

4. The following object plays a key role in our propositions. Define

$$\hat{\alpha}=1-\frac{\varepsilon\left(\mu-1\right)-\frac{\left(\mu^{\varepsilon}-1\right)}{\mu^{\varepsilon-1}}}{\left(\mu^{\varepsilon}-1\right)-\varepsilon\left(\mu-1\right)} \tag{A.5}$$

It can be shown that

$$\left(\mu^{\varepsilon}-1\right)-\varepsilon\left(\mu-1\right)>0, \tag{A.6}$$

since this expression equals 0 when $\mu=1$, and is increasing in μ. Similar arguments prove

$$\varepsilon\left(\mu-1\right)-\frac{\left(\mu^{\varepsilon}-1\right)}{\mu^{\varepsilon-1}}>0 \tag{A.7}$$

and (using the second derivative),

$$\left(\mu^{\varepsilon}-1\right)-\varepsilon\left(\mu-1\right)>\varepsilon\left(\mu-1\right)-\frac{\left(\mu^{\varepsilon}-1\right)}{\mu^{\varepsilon-1}}. \tag{A.8}$$

Hence $\hat{\alpha}\in(0,1)$.

5. From the results in 2., we obtain

$$\frac{\frac{\partial r(\alpha,\beta)}{\partial\alpha}}{r(\alpha,\beta)}-\frac{\frac{\partial r(\alpha,1)}{\partial\alpha}}{r(\alpha,1)}=\frac{(1-\beta)\left(1-\beta\,(1-\alpha)^{2}\,\mu^{\varepsilon-1}\right)\left(\mu^{\varepsilon-1}-1\right)}{(1+\beta\,(1-\alpha))\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon-1}\right)\left(1+(1-\alpha)\,\mu^{\varepsilon-1}\right)(2-\alpha)}. \tag{A.9}$$

6. The following derivation, which uses the derivatives of $r\,()$ and $g\,()$, will be useful below:

$$\begin{aligned}
\frac{\frac{\partial r(\alpha,\beta)}{\partial\alpha}}{r\,(\alpha,\beta)}-(\varepsilon-1)\,\frac{\frac{\partial g(\alpha,\beta)}{\partial\alpha}}{g\,(\alpha,\beta)} \quad &= \quad \frac{-\beta\left(\mu^{\varepsilon-1}-1\right)}{\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon-1}\right)(1+\beta\,(1-\alpha))}- \\
&\quad\ (\varepsilon-1)\,\frac{-\beta\mu^{\varepsilon-1}\,(\mu-1)}{\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon-1}\right)\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon}\right)} \\
&= \quad (\varepsilon-1)\,\frac{\beta\mu^{\varepsilon-1}\,(\mu-1)}{\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon-1}\right)\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon}\right)}- \\
&\quad\ \frac{\beta\left(\mu^{\varepsilon-1}-1\right)}{\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon-1}\right)(1+\beta\,(1-\alpha))} \\
&= \quad \left[\frac{\beta\mu^{\varepsilon-1}}{\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon-1}\right)\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon}\right)(1+\beta\,(1-\alpha))}\right] \\
&\quad\ \left[(\mu-1)\,(\varepsilon-1)\,(1+\beta\,(1-\alpha))-\frac{\left(\mu^{\varepsilon-1}-1\right)\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon}\right)}{\mu^{\varepsilon-1}}\right] \\
&= \quad \left[\frac{\beta\mu^{\varepsilon-1}}{\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon-1}\right)\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon}\right)(1+\beta\,(1-\alpha))}\right]\cdot \\
&\quad\ \left[\beta\,(1-\alpha)\left[(\mu-1)\,(\varepsilon-1)-\frac{\left(\mu^{\varepsilon-1}-1\right)\mu^{\varepsilon}}{\mu^{\varepsilon-1}}\right]+\right. \\
&\quad\ \left.(\mu-1)\,(\varepsilon-1)-\frac{\left(\mu^{\varepsilon-1}-1\right)}{\mu^{\varepsilon-1}}\right] \\
&= \quad \left[\frac{\beta\mu^{\varepsilon-1}}{\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon-1}\right)\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon}\right)(1+\beta\,(1-\alpha))}\right]\cdot \\
&\quad\ \left[\beta\,(1-\alpha)\left[\varepsilon\,(\mu-1)-(\mu-1)-\frac{\left(\mu^{\varepsilon-1}-1\right)\mu^{\varepsilon}}{\mu^{\varepsilon-1}}\right]+\right. \\
&\quad\ \left.\varepsilon\,(\mu-1)-(\mu-1)-\frac{\left(\mu^{\varepsilon-1}-1\right)}{\mu^{\varepsilon-1}}\right] \\
&= \quad \left[\frac{\beta\mu^{\varepsilon-1}}{\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon-1}\right)\left(1+\beta\,(1-\alpha)\,\mu^{\varepsilon}\right)(1+\beta\,(1-\alpha))}\right]\cdot \\
&\quad\ \left[\beta\,(1-\alpha)\left[\varepsilon\,(\mu-1)-\frac{\mu^{\varepsilon-1}\,(\mu-1)+\left(\mu^{\varepsilon-1}-1\right)\mu^{\varepsilon}}{\mu^{\varepsilon-1}}\right]+\right. \\
&\quad\ \left.\varepsilon\,(\mu-1)-\frac{\mu^{\varepsilon-1}\,(\mu-1)+\left(\mu^{\varepsilon-1}-1\right)}{\mu^{\varepsilon-1}}\right]
\end{aligned}$$

That is

$$\frac{\frac{\partial r(\alpha,\beta)}{\partial \alpha}}{r(\alpha,\beta)}-(\varepsilon-1)\frac{\frac{\partial g(\alpha,\beta)}{\partial \alpha}}{g(\alpha,\beta)}= \tag{A.10}$$

$$\beta\left[\frac{\mu^{\varepsilon-1}\left[(\mu^{\varepsilon}-1)-\varepsilon(\mu-1)\right]}{(1+\beta(1-\alpha)\mu^{\varepsilon-1})(1+\beta(1-\alpha)\mu^{\varepsilon})(1+\beta(1-\alpha))}\right]\cdot\left[(1-\hat{\alpha})-\beta(1-\alpha)\right]. \tag{A.11}$$

7. For convenience, we reproduce the following definitions here:

$$\mathsf{v}(\alpha;s(\bar{\alpha}))=\frac{\pi_{SUM}(\alpha,s(\bar{\alpha}))-C_{SUM}(\alpha)}{(1-\beta)} \tag{A.12}$$

$$\pi_{SUM}(\alpha,s(\bar{\alpha}))\equiv\left(\frac{1}{\varepsilon}\right)\left(\frac{D(\alpha,\beta)}{D(\bar{\alpha},\beta)}\right)^{\varepsilon-1}\left(\frac{r(\alpha;\beta)}{r(\alpha;1)}\right); \tag{A.13}$$

$$C_{SUM}(\alpha)\equiv\frac{\beta\chi\left[\alpha E\left(\xi|\xi<F^{-1}(\alpha)\right)+\beta(1-\alpha)E(\xi)\right]}{(1+\beta(1-\alpha))} \tag{A.14}$$

$$D(\alpha,\beta)\equiv\left(\frac{\varepsilon-1}{\varepsilon\chi}\right)\cdot\frac{r(\alpha;1)^{1/(\varepsilon-1)}}{g(\alpha;\beta)}. \tag{A.15}$$

A.2. Proofs

Lemma 3.3. For $\alpha\in[0,1]$, (i) When β is sufficiently small, $\frac{\partial D(\alpha,\beta)}{\partial\alpha}<0$; (ii) When β is sufficiently large, there exists $\tilde{\alpha}\in(0,1)$ – a function of β – such that $\frac{\partial D(\alpha,\beta)}{\partial\alpha}<0$ for $\alpha<\tilde{\alpha}$ and $\frac{\partial D(\alpha,\beta)}{\partial\alpha}>0$ for $\alpha>\tilde{\alpha}$. (iii) When β is sufficiently large, $D(\alpha,\beta)$ attains its maximum on $[0,1]$ at $\alpha=1$.

Proof of lemma 3.3: By differentiation of $D(\alpha,\beta)$,

$$\begin{aligned}
\frac{\frac{dD(\alpha,\beta)}{d\alpha}}{D(\alpha,\beta)} &= \left(\frac{1}{\varepsilon-1}\right)\left(\frac{\frac{\partial r(\alpha,1)}{\partial\alpha}}{r(\alpha,1)}\right)-\frac{\frac{\partial g(\alpha;\beta,\varepsilon,\mu)}{\partial\alpha}}{g(\alpha;\beta,\varepsilon,\mu)}\\
&= \left(\frac{1}{\varepsilon-1}\right)\left[\left(\frac{\frac{\partial r(\alpha,1)}{\partial\alpha}}{r(\alpha,1)}\right)-(\varepsilon-1)\frac{\frac{\partial g(\alpha;\beta,\varepsilon,\mu)}{\partial\alpha}}{g(\alpha;\beta,\varepsilon,\mu)}\right]\\
&= \left(\frac{1}{\varepsilon-1}\right)\left[\left\{\left(\frac{\frac{\partial r(\alpha,\beta)}{\partial\alpha}}{r(\alpha,\beta)}\right)-(\varepsilon-1)\frac{\frac{\partial g(\alpha;\beta,\varepsilon,\mu)}{\partial\alpha}}{g(\alpha;\beta,\varepsilon,\mu)}\right\}-\left\{\left(\frac{\frac{\partial r(\alpha,\beta)}{\partial\alpha}}{r(\alpha,\beta)}\right)-\left(\frac{\frac{\partial r(\alpha,1)}{\partial\alpha}}{r(\alpha,1)}\right)\right\}\right].
\end{aligned}$$

Now, use (A.10) together with (A.9) :

$$\begin{aligned}
\frac{\frac{dD(\alpha,\beta)}{d\alpha}}{D(\alpha,\beta)} = & \left(\frac{1}{\varepsilon-1}\right)\left\{\left(\frac{\beta\mu^{\varepsilon-1}\left[(\mu^{\varepsilon}-1)-\varepsilon(\mu-1)\right]}{(1+\beta(1-\alpha)\mu^{\varepsilon-1})(1+\beta(1-\alpha)\mu^{\varepsilon})(1+\beta(1-\alpha))}\right)\cdot\right.\\
& \left[(1-\hat{\alpha})-\beta(1-\alpha)\right] \qquad \text{(A.16)}\\
& \left.-\frac{(1-\beta)\left(1-\beta(1-\alpha)^{2}\mu^{\varepsilon-1}\right)(\mu^{\varepsilon-1}-1)}{(1+\beta(1-\alpha))(1+\beta(1-\alpha)\mu^{\varepsilon-1})(1+(1-\alpha)\mu^{\varepsilon-1})(2-\alpha)}\right\}
\end{aligned}$$

As $\beta \to 0$, the first term in (A.16) tends to 0, and the second term is negative, so $\frac{dD(\alpha,\beta)}{d\alpha}/D(\alpha,\beta) < 0 \forall\alpha \in [0,1]$. This proves part (i) of the lemma.

To prove part (ii), note that when $\beta = 1$, the second term in (A.16) equals 0, and so $\frac{dD(\alpha,\beta)}{d\alpha}/D(\alpha,\beta) < 0$ for $\alpha < \hat{\alpha}$ and $\frac{dD(\alpha,\beta)}{d\alpha}/D(\alpha,\beta) > 0$ for $\alpha > \hat{\alpha}$. Furthermore, this term is continuous in β. Hence for β sufficiently close to 1, there exists $\tilde{\alpha} \in (0,1)$ such that $\frac{dD(\alpha,\beta)}{d\alpha}/D(\tilde{\alpha},\beta) = 0$; $\tilde{\alpha}$ is implicitly defined by setting the entire expression in (A.16) equal to zero.

To prove part (iii) it is sufficient to show that $D(1,\beta) > D(0,\beta)$, since the result then follows immediately from part (ii). It is easily shown that this condition is equivalent to

$$\frac{1+\beta\mu^{\varepsilon}}{1+\beta\mu^{\varepsilon-1}} > \left[\frac{1+\mu^{\varepsilon-1}}{2}\right]^{\frac{1}{\varepsilon-1}}. \tag{A.17}$$

At $\beta = 1$, (A.17) becomes

$$\frac{1+\mu^{\varepsilon}}{1+\mu^{\varepsilon-1}} > \left[\frac{1+\mu^{\varepsilon-1}}{2}\right]^{\frac{1}{\varepsilon-1}}. \tag{A.18}$$

Multiplying both sides of (A.18) by $1+\mu^{\varepsilon-1}$, raising both sides to the power $\frac{1}{\varepsilon}$, and then taking logarithms, (A.18) is equivalent to

$$\left(\frac{1}{\varepsilon}\right)\ln\left(1+\mu^{\varepsilon}\right) > \left(\frac{1}{\varepsilon\left(1-\varepsilon\right)}\right)\ln 2 + \left(\frac{1}{\varepsilon-1}\right)\ln\left(1+\mu^{\varepsilon-1}\right). \tag{A.19}$$

We want to show that (A.19) holds for $\mu > 1$.When $\mu = 1$, the left hand and right hand sides of (A.19) are equal. Thus, it will be sufficient to show that the derivative of the left hand side with respect to μ exceeds the derivative of the right hand side, which means showing that

$$\left(\frac{\mu^{\varepsilon-1}}{1+\mu^{\varepsilon}}\right) > \left(\frac{\mu^{\varepsilon-2}}{1+\mu^{\varepsilon-1}}\right). \tag{A.20}$$

Multiplying both sides by $\left(1+\mu^{\varepsilon}\right)\left(1+\mu^{\varepsilon-1}\right)$ reveals that (A.20) holds for $\mu > 1$.We have thus shown that when $\mu > 1$ and $\beta = 1$, $D(1,\beta) > D(0,\beta)$. Because $D\left(\right)$ is continuous in β, this condition also holds for β close to 1. □

Lemma 3.4. (i) For small β, the interior arm of the best-response correspondence exhibits complementarity everywhere. (ii) As $\beta \to 1$, the interior arm of the best-response correspondence does not exhibit complementarity at any fixed point; (iii) As $\beta \to 1$, the interior arm of the best-response correspondence has a unique fixed point.

Proof of lemma 3.4: As a preliminary to proving the three parts of this proposition, we will derive a simple expression for the sign of the slope of the interior arm. By the implicit function theorem, the slope of the interior arm (3.31) is given by

$$\frac{\partial \alpha^*}{\partial \bar{\alpha}} = -\frac{\frac{\partial^2 \mathsf{v}(\alpha^*; s(\bar{\alpha}))}{\partial \alpha \partial \bar{\alpha}}}{\frac{\partial^2 \mathsf{v}(\alpha^*; s(\bar{\alpha}))}{\partial \alpha^2}} \tag{A.21}$$

and thus has the same sign as $\partial^2 \mathsf{v}(\alpha^*; s(\bar{\alpha})) / \partial \alpha \partial \bar{\alpha}$, since the denominator is negative by the second-order condition. Now from (A.12) and (A.13),

$$\frac{\partial^2 \mathsf{v}(\alpha^*; s(\bar{\alpha}))}{\partial \alpha \partial \bar{\alpha}} = \frac{1}{(1-\beta)} \frac{\partial^2 \pi_{SUM}(\alpha, s(\bar{\alpha}))}{\partial \alpha \partial \bar{\alpha}}$$

because C_{SUM} is independent of $\bar{\alpha}$, so

$$sgn\left\{\frac{\partial \alpha^*}{\partial \bar{\alpha}}\right\} = sgn\left\{\frac{\partial^2 \pi_{SUM}(\alpha, s(\bar{\alpha}))}{\partial \alpha \partial \bar{\alpha}}\right\} \tag{A.22}$$

That is, the existence or absence of complementarity hinges on the cross-partial of the profit sum with respect to α and $\bar{\alpha}$. In turn,

$$\frac{\partial \pi_{SUM}(\alpha, s(\bar{\alpha}))}{\partial \alpha} = \pi_{SUM}(\alpha, s(\bar{\alpha})) \cdot \left[(\varepsilon - 1)\frac{\frac{\partial D(\alpha,\beta)}{\partial \alpha}}{D(\alpha,\beta)} + \frac{\frac{\partial r(\alpha,\beta)}{\partial \alpha}}{r(\alpha,\beta)} - \frac{\frac{\partial r(\alpha,1)}{\partial \alpha}}{r(\alpha,1)}\right] \tag{A.23}$$

To prove part (i), note that it follows that

$$\begin{aligned}
\frac{\partial \pi_{SUM}(\alpha, s(\bar{\alpha}))}{\partial \alpha} &= \pi_{SUM}(\alpha, s(\bar{\alpha})) \cdot \left[\frac{\frac{\partial r(\alpha,1)}{\partial \alpha}}{r(\alpha,1)} - (\varepsilon - 1)\frac{\frac{\partial g(\alpha,\beta)}{\partial \alpha}}{g(\alpha,\beta)} + \frac{\frac{\partial r(\alpha,\beta)}{\partial \alpha}}{r(\alpha,\beta)} - \frac{\frac{\partial r(\alpha,1)}{\partial \alpha}}{r(\alpha,1)}\right] \\
&= \pi_{SUM}(\alpha, s(\bar{\alpha})) \cdot \left[\frac{\frac{\partial r(\alpha,\beta)}{\partial \alpha}}{r(\alpha,\beta)} - (\varepsilon - 1)\frac{\frac{\partial g(\alpha,\beta)}{\partial \alpha}}{g(\alpha,\beta)}\right]
\end{aligned}$$

which, from (A.10), implies

$$\begin{aligned}
\frac{\partial \pi_{SUM}(\alpha, s(\bar{\alpha}))}{\partial \alpha} &= \pi_{SUM}(\alpha, s(\bar{\alpha}))\left(\frac{\beta\mu^{\varepsilon-1}\left[(\mu^{\varepsilon}-1) - \varepsilon(\mu-1)\right]}{(1+\beta(1-\alpha)\mu^{\varepsilon-1})(1+\beta(1-\alpha)\mu^{\varepsilon})(1+\beta(1-\alpha))}\right) \\
&\quad \times \left[(1-\hat{\alpha}) - \beta(1-\alpha)\right] .
\end{aligned} \tag{A.24}$$

For β sufficiently small, the term in square brackets is positive. Notice also that $\bar{\alpha}$ shows up only in $\pi_{SUM}(\alpha, s(\bar{\alpha}))$, and that $\pi_{SUM}(\alpha, s(\bar{\alpha}))$ is decreasing in $D(\bar{\alpha}, \beta)$. Finally, recall from Lemma 3.3, that for β sufficiently small, $\partial D(\bar{\alpha}, \beta) / \partial \bar{\alpha} < 0$. The result follows.

To prove (ii), note that as $\beta \to 1$, the final two terms in (A.23) cancel, so that

$$\frac{\partial \pi_{SUM}(\alpha, s(\bar{\alpha}))}{\partial \alpha} \to \pi_{SUM}(\alpha, s(\bar{\alpha})) \cdot \left[(\varepsilon - 1)\frac{\frac{\partial D(\alpha,1)}{\partial \alpha}}{D(\alpha,1)}\right]$$

and so

$$\begin{aligned}
\frac{\partial^2 \pi_{SUM}(\alpha, s(\bar{\alpha}))}{\partial \alpha \partial \bar{\alpha}} &\to \frac{\partial \pi_{SUM}(\alpha, s(\bar{\alpha}))}{\partial \bar{\alpha}} \cdot \left[(\varepsilon - 1) \frac{\frac{\partial D(\alpha,1)}{\partial \alpha}}{D(\alpha, 1)} \right] \\
&= \frac{-(\varepsilon - 1)^2 \cdot \pi_{SUM}(\alpha, s(\bar{\alpha}))}{D(\alpha, 1) \cdot D(\bar{\alpha}, 1)} \cdot \left[\frac{\partial D(\alpha, 1)}{\partial \alpha} \right] \cdot \left[\frac{\partial D(\bar{\alpha}, 1)}{\partial \bar{\alpha}} \right].
\end{aligned}$$

Thus for $\alpha = \bar{\alpha}$, this cross-partial is negative.

(iii) A necessary (not sufficient) condition for multiple fixed points of the interior arm is that the best-response function cuts the 45 degree line with positive slope. Part (ii) of the lemma establishes that this is impossible when $\beta = 1$. In the vicinity of $\beta = 1$, it follows from (A.24) that

$$\begin{aligned}
sgn\left\{ \frac{\partial \alpha^*}{\partial \bar{\alpha}} \right\} &= sgn\left\{ \frac{\partial^2 \pi_{SUM}(\alpha, s(\bar{\alpha}))}{\partial \alpha \partial \bar{\alpha}} \right\} = sgn\left\{ \left(\frac{\partial \pi_{SUM}(\alpha, s(\bar{\alpha}))}{\partial \bar{\alpha}} \right) \cdot [(1 - \hat{\alpha}) - \beta(1 - \alpha)] \right\} \\
&= sgn\left\{ - \left[\left(\frac{\partial D(\bar{\alpha}, \beta)}{\partial \bar{\alpha}} \right) [(1 - \hat{\alpha}) - \beta(1 - \alpha)] \right] \right\} \\
&= sgn\left\{ \left(\frac{\partial D(\bar{\alpha}, \beta)}{\partial \bar{\alpha}} \right) \cdot [\check{\alpha} - \alpha] \right\}, \qquad \text{(A.25)}
\end{aligned}$$

where

$$\check{\alpha} \equiv \frac{\hat{\alpha} - (1 - \beta)}{\beta}. \qquad \text{(A.26)}$$

Here, $\check{\alpha}$ defines the value of α where $\partial \pi_{SUM}(\alpha, s(\bar{\alpha})) / \partial \alpha$ changes sign. For β close to 1, from Lemma 3.3 we know that $\partial D(\bar{\alpha}, \beta) / \partial \bar{\alpha} < 0$ for $\bar{\alpha} < \tilde{\alpha}$ and $\partial D(\bar{\alpha}, \beta) / \partial \bar{\alpha} > 0$ for $\bar{\alpha} > \tilde{\alpha}$. Trivially, $\check{\alpha} - \alpha$ is positive for $\alpha < \check{\alpha}$, and negative for $\alpha > \check{\alpha}$. Thus, from (A.25) the interior arm slopes down if $\alpha < \check{\alpha}$ and $\bar{\alpha} < \tilde{\alpha}$, or if $\alpha > \check{\alpha}$ and $\bar{\alpha} > \tilde{\alpha}$. At a fixed point, $\alpha = \bar{\alpha} = \alpha^{**}$. Define $\Lambda = [\min(\check{\alpha}, \tilde{\alpha}), \max(\check{\alpha}, \tilde{\alpha})]$. The interior arm slopes down at α^{**} unless $\alpha^{**} \in \Lambda$. Now, $\check{\alpha}$ and $\tilde{\alpha}$ vary continuously in β, and as $\beta \to 1, \check{\alpha} \to \hat{\alpha}$ and $\tilde{\alpha} \to \hat{\alpha}$. Hence, as $\beta \to 1$, $\Lambda \to \{\emptyset\}$. For $\beta \simeq 1$, Λ is non-empty, and so if there exists $\alpha^{**} \in \Lambda$, then the best-response function is upward sloping when it cuts the 45 degree line. Note, however, that Λ is independent of the menu cost distribution, whereas the position of the best-response function depends upon $C_{SUM}(\alpha)$ and so varies continuously with $F(\cdot)$. As $\beta \to 1$, $\Lambda \to \{\emptyset\}$, and so, for almost every menu cost distribution $F(\cdot)$, the best-response function will not have a fixed point in Λ. $\square$

As noted in the text, the following proposition considers the case where there are two local maxima, one of which is the flexible arm. The results that we obtain for this case can be modified to apply to discontinuities between any two local maxima, although the sufficient conditions that we derive would differ. The best-response correspondence is given by

$$\alpha = \begin{cases} \alpha^*(\bar{\alpha}) \text{ if } \mathsf{v}(\alpha^*;s(\bar{\alpha})) \geq \mathsf{v}(1;s(\bar{\alpha})) \\ 1 \text{ if } \mathsf{v}(1;s(\bar{\alpha})) \geq \mathsf{v}(\alpha^*;s(\bar{\alpha})) \end{cases}.$$

Proposition 3.5. Let β be sufficiently large that the interior arm has a unique fixed point (see Lemma 3.4). Denote this fixed point by α^{**}. Let $\hat{\alpha}$ be as defined above. As $\beta \to 1$, necessary conditions for multiple equilibria are:

(i) $\alpha^{**} < \hat{\alpha}$

(ii) $\mathsf{v}(\alpha^*(\hat{\alpha}), s(\hat{\alpha})) < \mathsf{v}(1, s(\hat{\alpha}))$.

Proof of Proposition 3.5: From Lemma 3.4, we know that, as $\beta \to 1$, there cannot be multiple interior equilibria. Thus multiple equilibria are possible only if the best-response correspondence possesses discontinuities. Furthermore, multiplicity requires that there be a discontinuity that jumps over the 45-degree line from below. The proposition looks for conditions such that such jumps do not occur.

First we note that for β sufficiently large, $D(1,\beta) > D(\alpha,\beta) \forall \alpha \in [0,1)$, and therefore

$$D(1,\beta) > D(\alpha^*(\bar{\alpha}),\beta). \tag{A.27}$$

This follows immediately from parts (ii) and (iii) of Lemma 3.3.

Second, we recall from Lemma 3.3 that for $\bar{\alpha} \in (\tilde{\alpha}, 1]$, $dD(\bar{\alpha},\beta)/d\bar{\alpha} > 0$, and for $\bar{\alpha} \in [0,\tilde{\alpha})$, $dD(\bar{\alpha},\beta)/d\bar{\alpha} < 0$, where $\tilde{\alpha}$ depends on β and is defined in Lemma 3.3.

We now show that for $\bar{\alpha} \in [\tilde{\alpha}, 1]$, any discontinuities involve downward jumps – that is, the global maximum shifts from the flexible arm to the interior arm. Recall that

$$\pi_{SUM}(\alpha, s(\bar{\alpha})) = \left(\frac{1}{\varepsilon}\right)\left(\frac{D(\alpha,\beta)}{D(\bar{\alpha},\beta)}\right)^{\varepsilon-1}\left(\frac{r(\alpha,\beta)}{r(\alpha,1)}\right)$$

and that C_{SUM} is independent of $\bar{\alpha}$. It follows that

$$\frac{\partial \mathsf{v}(1, s(\bar{\alpha}))}{\partial \bar{\alpha}} = -\left(\frac{\varepsilon-1}{\varepsilon(1-\beta)}\right)\left(\frac{D(1,\beta)}{D(\bar{\alpha},\beta)}\right)^{\varepsilon-1}\left(\frac{r(1,\beta)}{r(1,1)}\right)\frac{\frac{dD(\bar{\alpha},\beta)}{d\bar{\alpha}}}{D(\bar{\alpha},\beta)} < 0 \; for \; \bar{\alpha} \in [\tilde{\alpha}, 1]$$

since $\frac{\frac{dD(\bar{\alpha},\beta)}{d\bar{\alpha}}}{D(\bar{\alpha},\beta)} > 0$ in this range. Also

$$\frac{\partial \mathsf{v}(\alpha^*(\bar{\alpha}), s(\bar{\alpha}))}{\partial \bar{\alpha}} = -\left(\frac{\varepsilon-1}{\varepsilon(1-\beta)}\right)\left(\frac{D(\alpha^*(\bar{\alpha}),\beta)}{D(\bar{\alpha},\beta)}\right)^{\varepsilon-1}\left(\frac{r(\alpha^*(\bar{\alpha}),\beta)}{r(\alpha^*(\bar{\alpha}),1)}\right)\frac{\frac{dD(\bar{\alpha},\beta)}{d\bar{\alpha}}}{D(\bar{\alpha},\beta)} < 0 \; for \; \bar{\alpha} \in [\tilde{\alpha}, 1].$$

In the second expression we use the envelope theorem to ignore changes in α^*. Now, it is easily confirmed that $\left(\frac{r(1,\beta)}{r(1,1)}\right) > \left(\frac{r(\alpha,\beta)}{r(\alpha,1)}\right)$.[32] Meanwhile, we have already established that $D(1,\beta) > D(\alpha,\beta)$. Hence

$$\left|\frac{\partial \mathsf{v}(1, s(\bar{\alpha}))}{\partial \bar{\alpha}}\right| > \left|\frac{\partial \mathsf{v}(\alpha^*(\bar{\alpha}), s(\bar{\alpha}))}{\partial \bar{\alpha}}\right| \quad for\ \bar{\alpha} \in [\tilde{\alpha}, 1]$$

This means that the value function is decreasing more rapidly at $\alpha = 1$ (the flexible arm) than at $\alpha = \alpha^*$. Now suppose that $\mathsf{v}(\alpha^*(\tilde{\alpha}), s(\tilde{\alpha})) > \mathsf{v}(1, s(\tilde{\alpha}))$, meaning that, at $\tilde{\alpha}$, the best-response function is on the interior arm at this point. Then it follows that $\mathsf{v}(\alpha^*(\tilde{\alpha}), s(\bar{\alpha})) > \mathsf{v}(1, s(\bar{\alpha}))$ for all $\bar{\alpha} \in [\tilde{\alpha}, 1]$. The best-response function will not jump to the flexible arm above $\tilde{\alpha}$. It follows that there are no multiplicities for $\bar{\alpha} \in [\tilde{\alpha}, 1]$.

Similar reasoning shows that for $\bar{\alpha} \in [0, \tilde{\alpha}]$ any discontinuities involve upward jumps. Multiple equilibria are thus only possible if a discontinuity occurs for some $\bar{\alpha} \in [\alpha^{**}, \tilde{\alpha}]$, because multiplicity requires the best response correspondence to cross the 45 degree line on the interior arm and then jump up to the flexible arm.

Thus, necessary conditions for multiple equilibria are:

(i) $\alpha^{**} < \tilde{\alpha}$

(ii) $\mathsf{v}(\alpha^*(\tilde{\alpha}), s(\alpha^{**})) < \mathsf{v}(1, s(\alpha^{**}))$.

The first condition states that the fixed point of the interior arm is below $\tilde{\alpha}$. If this is not satisfied, upward jumps are not possible, as shown above. The second condition states that there is a discontinuity for some $\bar{\alpha} \in [\alpha^{**}, \tilde{\alpha}]$. Finally, we have already shown that as $\beta \to 1$, $\tilde{\alpha} \to \hat{\alpha}$, which completes the proof. $\square$

Corollary 3.6, part (a). As $\beta \to 1$, there is a unique equilibrium if

$$\left(\frac{1}{\varepsilon}\right)\left[\left(\frac{1+\mu^{\varepsilon}}{1+\mu^{\varepsilon-1}}\right)^{\varepsilon}\left(\frac{2}{1+\mu^{\varepsilon}}\right) - 1\right] > \chi E(\xi).$$

Proof: If $\alpha^*(0) > \hat{\alpha}$ then $\alpha^{**} > \hat{\alpha}$ in contradiction of (i). Therefore, assume $\alpha^*(0) < \hat{\alpha}$.

[32] Note also that as $\beta \to 1$, $\left(\frac{r(\alpha^*(\bar{\alpha}),\beta)}{r(\alpha^*(\bar{\alpha}),1)}\right) \to 1$, and so can be neglected.

The condition above is a rewriting of

$$\begin{aligned}
\left(\frac{1}{\varepsilon}\right)\left[\left(\frac{D(1,1)}{D(0,1)}\right)^{\varepsilon-1}-1\right] &> \chi E(\xi) \\
&\Rightarrow \\
\left(\frac{1}{\varepsilon}\right)\left[\left(\frac{D(1,1)}{D(0,1)}\right)^{\varepsilon-1}-\left(\frac{D(0,1)}{D(0,1)}\right)^{\varepsilon-1}\right] &> \chi E(\xi) \\
&\Rightarrow \\
\left(\frac{1}{\varepsilon}\right)\left[\left(\frac{D(1,1)}{D(0,1)}\right)^{\varepsilon-1}-\left(\frac{D(\alpha^*(0),1)}{D(0,1)}\right)^{\varepsilon-1}\right] &> \chi E(\xi) \\
&\Rightarrow \\
\left(\frac{1}{\varepsilon}\right)\left[\left(\frac{D(1,1)}{D(0,1)}\right)^{\varepsilon-1}-\left(\frac{D(\alpha^*(0),1)}{D(0,1)}\right)^{\varepsilon-1}\right] &> \chi E(\xi) - C_{SUM}(\alpha^*(0)) \\
&\Rightarrow \\
\left(\frac{1}{\varepsilon}\right)\left[\left(\frac{D(1,1)}{D(0,1)}\right)^{\varepsilon-1}-\left(\frac{D(\alpha^*(0),1)}{D(0,1)}\right)^{\varepsilon-1}\right] &> C_{SUM}(1) - C_{SUM}(\alpha^*(0))
\end{aligned}$$

These last conditions follow because $D(\alpha^*(0),1) < D(0,1)$ (since $D(\alpha,\beta)$ is decreasing in α for $\alpha < \hat{\alpha}$), because $C_{SUM}(\alpha^*(0)) > 0$, and because $\chi E(\xi) = C_{SUM}(1)$. Now recall that

$$\begin{aligned}
\mathsf{v}(\alpha; s(\bar{\alpha})) &= \frac{\pi_{SUM}(\alpha, s(\bar{\alpha})) - C_{SUM}(\alpha)}{(1-\beta)} \\
&\Rightarrow (1-\beta)\,\mathsf{v}(\alpha; s(\bar{\alpha})) = \pi_{SUM}(\alpha, s(\bar{\alpha})) - C_{SUM}(\alpha) \\
&= \left(\frac{1}{\varepsilon}\right)\left(\frac{D(\alpha,\beta)}{D(\bar{\alpha},\beta)}\right)^{\varepsilon-1}\left(\frac{r(\alpha,\beta)}{r(\alpha,1)}\right) - C_{SUM}(\alpha)
\end{aligned}$$

As $\beta \to 1$, $\left(\frac{r(\alpha,\beta)}{r(\alpha,1)}\right) \to 1$, and the right-hand side of the above expression tends to $\left(\frac{1}{\varepsilon}\right)\left(\frac{D(\alpha,1)}{D(\bar{\alpha},1)}\right)^{\varepsilon-1} - C_{SUM}(\alpha)$. Thus the condition just derived is equivalent to

$$\mathsf{v}(1, s(0)) > \mathsf{v}(\alpha^*(0), s(0))$$

which implies that, at $\bar{\alpha} = 0$, the best-response function is on the flexible arm. Since downward jumps are not possible for $\bar{\alpha} \in [0, \hat{\alpha}]$, it follows that

$$\mathsf{v}(1, s(\alpha^{**})) > \mathsf{v}(\alpha^{**}, s(\alpha^{**})),$$

which says that the fixed point of the interior arm is not an equilibrium.

Corollary 3.6, part (b). As $\beta \to 1$, there is a unique equilibrium if

$$\chi E(\xi) - C_{SUM}(\hat{\alpha}) > \left(\frac{1}{\varepsilon}\right)\left[\frac{(\mu^{\varepsilon}-1)}{\varepsilon(\mu-1)\mu^{\varepsilon-1}} \cdot \left(\frac{(\mu^{\varepsilon}-1)(\varepsilon-1)}{(\mu^{\varepsilon-1}-1)\varepsilon}\right)^{\varepsilon-1} - 1\right]$$

Proof: This expression can be rewritten as

$$\chi E(\xi) - C_{SUM}(\hat{\alpha}) > \left(\frac{1}{\varepsilon}\right)\left[\left(\frac{D(1,1)}{D(\hat{\alpha},1)}\right)^{\varepsilon-1} - 1\right]$$

$$\Rightarrow \quad C_{SUM}(1) - C_{SUM}(\hat{\alpha}) > \left(\frac{1}{\varepsilon}\right)\left[\left(\frac{D(1,1)}{D(\hat{\alpha},1)}\right)^{\varepsilon-1} - \left(\frac{D(\hat{\alpha},1)}{D(\hat{\alpha},1)}\right)^{\varepsilon-1}\right]$$

By similar reasoning to the previous case, as $\beta \to 1$, this expression is equivalent to

$$\Rightarrow \quad \mathsf{v}(\hat{\alpha}, s(\hat{\alpha})) > \mathsf{v}(1, s(\hat{\alpha}))$$

$$\Rightarrow \quad \mathsf{v}(\alpha^*(\hat{\alpha}), s(\hat{\alpha})) > \mathsf{v}(1, s(\hat{\alpha}))$$

(since $\mathsf{v}(\alpha^*(\hat{\alpha}), s(\hat{\alpha})) > \mathsf{v}(\hat{\alpha}, s(\hat{\alpha}))$), which is in violation of condition (ii). $\square$

B. Deriving the Steady–State Value Function in the Yeoman-Farmer Model

We suppose that the yeoman farmer has preferences given by

$$u(c,n) = \ln c - \chi n^{\nu}.$$

The value function for a typical agent in the yeoman-farmer model is

$$V\left(x_t, \xi; s\right) = \max \begin{cases} \ln\left[\left(x_t y\right)^{1-\varepsilon} y\right] - \chi\left[\left(x_t y\right)^{-\varepsilon} y\right]^{\nu} + \beta E V\left(\frac{x_t}{\mu}, \xi'; s\right), \\ \max_{x_0}\left[\ln\left[\left(x_0 y\right)^{1-\varepsilon} y\right] - \chi\left[\left(x_0 y\right)^{-\varepsilon} y\right]^{\nu} - \chi\xi + \beta E V\left(\frac{x_0}{\mu}, \xi'; s\right)\right] \end{cases} \tag{B.1}$$

The first-order condition for the choice of x_0 is

$$\frac{1-\varepsilon}{x^*} + \varepsilon\nu\chi y^{\nu(1-\varepsilon)}\left(x^*\right)^{-1-\varepsilon\nu} + \beta\left(1-\alpha\right)\left[\frac{1-\varepsilon}{x^*} + \varepsilon\nu\chi y^{\nu(1-\varepsilon)}\mu^{\varepsilon\nu}\left(x^*\right)^{-1-\varepsilon\nu}\right] = 0,$$

which implies

$$x^* y = \left[\left(\frac{\varepsilon\nu\chi}{\varepsilon-1}\right)\cdot y^{\nu}\cdot q(\alpha,\beta;\varepsilon,\mu,\nu)\right]^{\frac{1}{\varepsilon\nu}}$$

where

$$q(\alpha,\beta;\varepsilon,\mu,\nu) = \left(\frac{1+\beta\left(1-\alpha\right)\mu^{\varepsilon\nu}}{1+\beta\left(1-\alpha\right)}\right).$$

The price index is unchanged from our previous analysis:

$$x^* = y^{-1}\cdot r(\bar{\alpha}, 1; \varepsilon, \mu)^{1/(\varepsilon-1)},$$

which means that we can write

$$y\left(\bar{\alpha}\right) = \left(\frac{\varepsilon-1}{\varepsilon\nu\chi}\right)^{\frac{1}{\nu}} r(\bar{\alpha}, 1; \varepsilon, \mu)^{\frac{\varepsilon}{\varepsilon-1}} q(\bar{\alpha}, \beta; \varepsilon, \mu)^{-\frac{1}{\nu}}.$$

We now derive the steady-state value function for an individual farmer. Let u^A denote the utility of a farmer who adjusts in the current period (gross of adjustment costs), and u^N

the utility of a farmer who does not adjust. A farmer who adjusts obtains

$$\begin{aligned}
u^A &= \ln\left((xy)^{1-\varepsilon}y\right) - \chi\left[(xy)^{-\varepsilon}y\right]^{\nu} \\
&= \frac{1-\varepsilon}{\varepsilon\nu}\left[\ln\left(\frac{\varepsilon\nu\chi}{\varepsilon-1}\right) + \nu\ln y(\bar{\alpha}) + \ln q(\alpha)\right] + \ln y(\bar{\alpha}) \\
&\quad -\chi\left[\left(\frac{\varepsilon\nu\chi}{\varepsilon-1}\right)\cdot y(\bar{\alpha})^{\nu}\cdot q(\alpha,\beta;\varepsilon,\mu,\nu)\right]^{-1}\cdot y(\bar{\alpha})^{\nu} \\
&= \frac{1-\varepsilon}{\varepsilon\nu}\left[\ln\left(\frac{\varepsilon\nu\chi}{\varepsilon-1}\right) + \nu\ln y(\bar{\alpha}) + \ln q(\alpha)\right] + \ln y(\bar{\alpha}) \\
&\quad -\left[\left(\frac{\varepsilon\nu}{\varepsilon-1}\right)\cdot q(\alpha,\beta;\varepsilon,\mu,\nu)\right]^{-1} \\
&= \left(\frac{1}{\varepsilon}\right)\ln y(\bar{\alpha}) + \frac{1-\varepsilon}{\varepsilon\nu}\left[\ln\left(\frac{\varepsilon\nu\chi}{\varepsilon-1}\right) + \ln q(\alpha)\right] \\
&\quad -\left[\left(\frac{\varepsilon\nu}{\varepsilon-1}\right)\cdot q(\alpha,\beta;\varepsilon,\mu,\nu)\right]^{-1}
\end{aligned}$$

and a farmer who does not adjust obtains

$$\begin{aligned}
u^N &= \left(\frac{1}{\varepsilon}\right)\ln y(\bar{\alpha}) + \frac{1-\varepsilon}{\varepsilon\nu}\left[\ln\left(\frac{\varepsilon\nu\chi}{\varepsilon-1}\right) + \ln q(\alpha)\right] \\
&\quad +(\varepsilon-1)\ln\mu - \mu^{\varepsilon\nu}\left[\left(\frac{\varepsilon\nu}{\varepsilon-1}\right)\cdot q(\alpha,\beta;\varepsilon,\mu,\nu)\right]^{-1}
\end{aligned}$$

The steady-state value function is

$$v(\alpha,\bar{\alpha}) = u^A + \beta\left\{(1-\alpha)\left[u^N + \beta\left(v(\alpha,\bar{\alpha}) - \chi\bar{\xi}\right)\right] + \alpha\left[v(\alpha,\bar{\alpha}) - \chi E\left(\xi|\xi < F^{-1}(\alpha)\right)\right]\right\}$$

$$\begin{aligned}
&\Rightarrow (1-\beta)v(\alpha,\bar{\alpha}) = \frac{u^A + \beta(1-\alpha)u^N - \beta\chi\left[\alpha E\left(\xi|\xi < F^{-1}(\alpha)\right) + \beta(1-\alpha)E\xi\right]}{(1+\beta(1-\alpha))} \\
&= \left(\frac{1}{\varepsilon}\right)\ln y(\bar{\alpha}) + \frac{1-\varepsilon}{\varepsilon}\left[\ln\left(\frac{\varepsilon\chi}{\varepsilon-1}\right) + \ln q(\alpha)\right] + \frac{\beta(1-\alpha)(\varepsilon-1)\ln\mu}{(1+\beta(1-\alpha))} \\
&\quad -\frac{(1+\beta(1-\alpha)\mu^{\varepsilon\nu})\chi\left[\left(\frac{\varepsilon\chi}{\varepsilon-1}\right)\cdot q(\alpha,\beta;\varepsilon,\mu,\nu)\right]^{-1}}{(1+\beta(1-\alpha))} - \frac{\beta\chi\left[\alpha E\left(\xi|\xi < F^{-1}(\alpha)\right) + \beta(1-\alpha)E\xi\right]}{(1+\beta(1-\alpha))} \\
&= \left(\frac{1}{\varepsilon}\right)\ln y(\bar{\alpha}) + \frac{1-\varepsilon}{\varepsilon\nu}\left[\ln\left(\frac{\varepsilon\nu\chi}{\varepsilon-1}\right) + \ln q(\alpha)\right] + \frac{\beta(1-\alpha)(\varepsilon-1)\ln\mu}{(1+\beta(1-\alpha))} \\
&\quad -\frac{\varepsilon-1}{\varepsilon\nu} - \frac{\beta\chi\left[\alpha E\left(\xi|\xi < F^{-1}(\alpha)\right) + \beta(1-\alpha)E\xi\right]}{(1+\beta(1-\alpha))}.
\end{aligned}$$

C. Constructing the Final Period Best-Response Function

In this appendix we show how to construct a best response function for price adjustment in the final period of a finite horizon model. We begin by collecting the relevant equations. They are the price index,

$$c_T^{\varepsilon-1} = \omega_{T+1}\left(x_{T+1}^*\right)^{1-\varepsilon} + \left(1-\omega_{T+1}\right)\left(x_T^*\right)^{1-\varepsilon}\mu_T^{\varepsilon-1} \tag{C.1}$$

labor supply,

$$w_T = \chi c_T \tag{C.2}$$

optimal pricing (static),

$$x_{T+1}^* = \frac{\varepsilon\chi}{\varepsilon-1} \tag{C.3}$$

final period profits of adjusters,

$$\pi_T^* = c_T^{2-\varepsilon}\left(\eta\chi\right)^{-\varepsilon}\left[\eta\chi-\chi\right], \tag{C.4}$$

final period profits of non-adjusters,

$$\pi_T^N = c_T^{2-\varepsilon}\left(x_T^*/\mu_T\right)^{-\varepsilon}\left[x_T^*/\mu_T-\chi\right] \tag{C.5}$$

the conditional adjustment probability in the final period,

$$\alpha_T = F\left(\frac{\pi^*-\pi_N}{w}\right), \tag{C.6}$$

and the fraction of firms adjusting,

$$\omega_{T+1} = \left(1-\omega_T\right) + \omega_T\cdot\bar{\alpha}_T. \tag{C.7}$$

Note that from (C.4) - (C.6),

$$\alpha_T = F\left(c_T^{1-\varepsilon}\cdot\left(\left(\eta\chi\right)^{-\varepsilon}\left(\frac{\varepsilon}{\varepsilon-1}-1\right)-\left(\frac{x_T^*}{\mu_T}\right)^{-\varepsilon}\left(\frac{x_T^*}{\mu_T\chi}-1\right)\right)\right). \tag{C.8}$$

From (C.1), (C.7), (C.3) and (C.6),

$$\begin{aligned}
\alpha_T &= F\left(\frac{G^2\left(\mathfrak{P}\right)}{G^1\left(\bar{\alpha}_T;\mathfrak{P}\right)}\right), \text{ where} \\
G^1\left(\alpha_T;\mathfrak{P}\right) &\equiv \left(\left(1-\omega_T\right)+\omega_T\cdot\bar{\alpha}_T\right)\left(\frac{\varepsilon\chi}{\varepsilon-1}\right)^{1-\varepsilon} \\
&\quad +\omega_T\cdot\left(1-\bar{\alpha}_T\right)\left(x_T^*\right)^{1-\varepsilon}\mu_T^{\varepsilon-1} \\
G^2\left(\mathfrak{P}\right) &\equiv \left(\eta\chi\right)^{-\varepsilon-\varepsilon}\left(\frac{1}{\varepsilon-1}\right)-\left(\frac{x_T^*}{\mu_T}\right)^{-\varepsilon}\left(\frac{x_T^*}{\mu_T\chi}-1\right).
\end{aligned} \tag{C.9}$$

Final period equilibria are fixed points of the response function (C.9). The question of complementarity is the question of whether the derivative of the right hand side of (C.9) is positive, which turns on whether $\frac{\partial G^1(\bar{\alpha}_T;\mathfrak{P})}{\partial \alpha_T} < 0$. It is straightforward to show that

$$sign\left(\frac{\partial G^1\left(\bar{\alpha}_T;\mathfrak{P}\right)}{\partial \alpha_T}\right) = sign\left(x_T^* - \frac{\varepsilon\chi}{\varepsilon - 1}\mu_T\right),$$

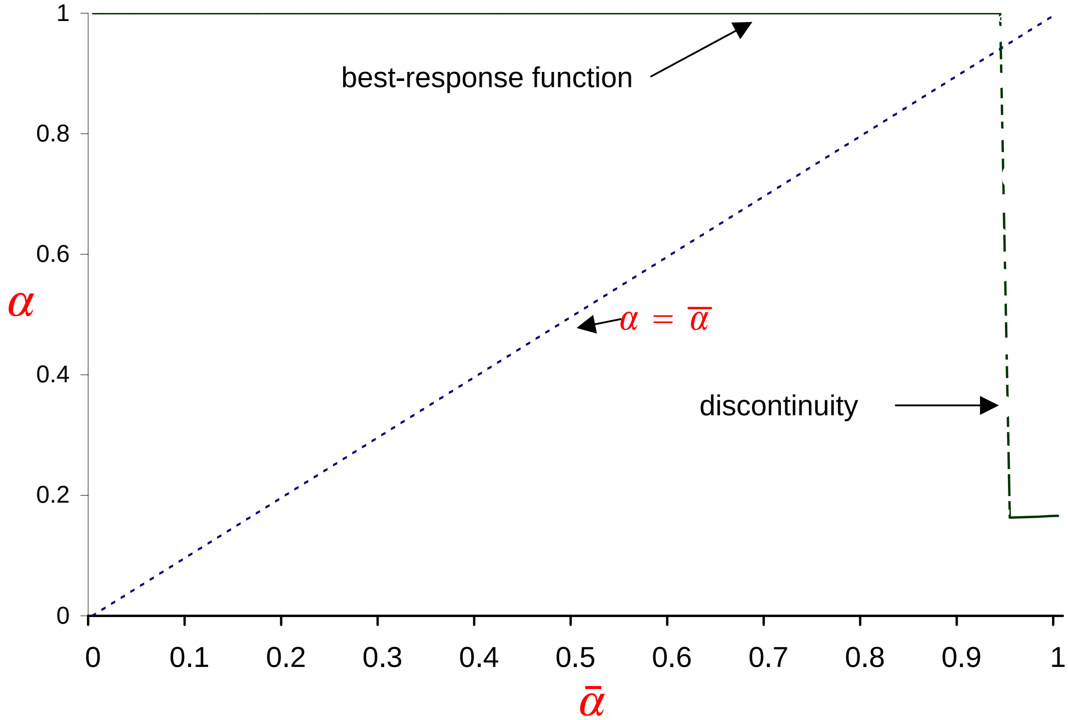
so that there is complementarity in price adjustment in the final period whenever

$$x_T^* < \frac{\varepsilon\chi}{\varepsilon - 1}\mu_T,$$

that is, whenever the monetary shock is sufficiently large.

Figure 1
Discontinuous best response.
Markup=1.1, inflation=1.085

A. Best-response function $\alpha(\overline{\alpha})$



B. v(α)

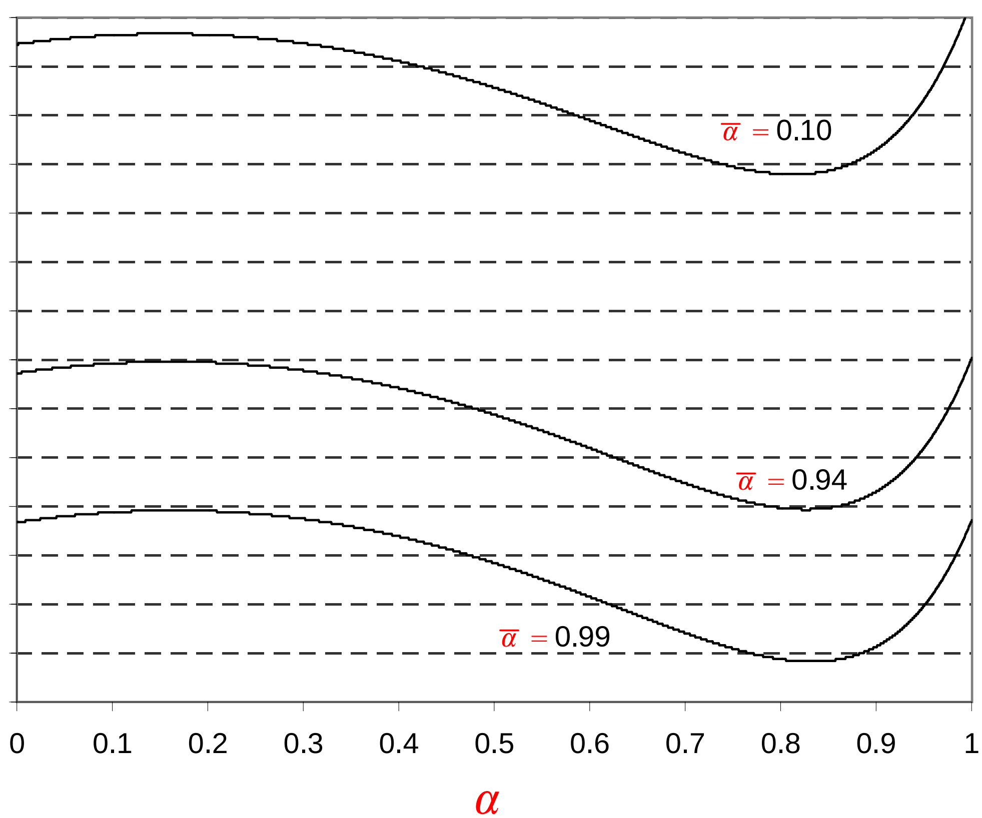


Figure 2: Hypothetical Best-Response Functions, β = 1

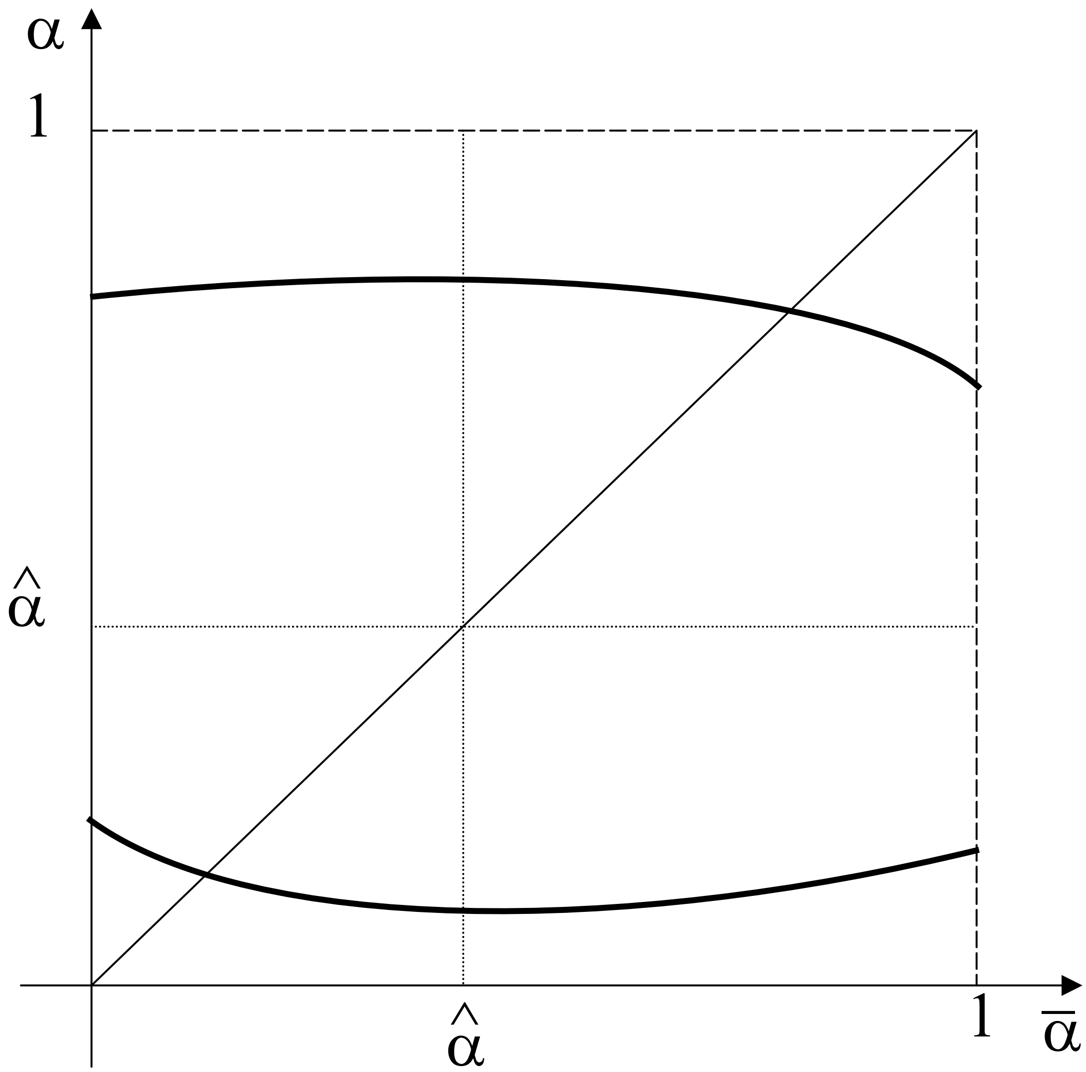


Figure 3. Best-response functions for different values of the discount factor

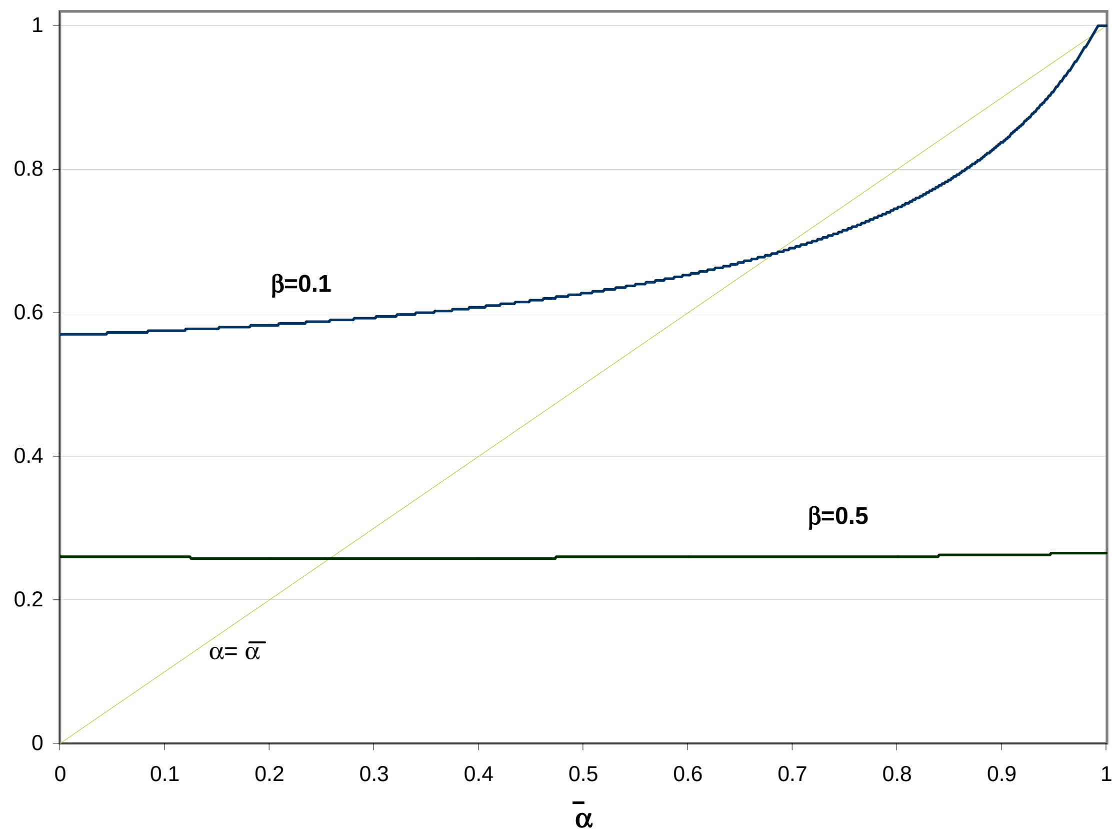
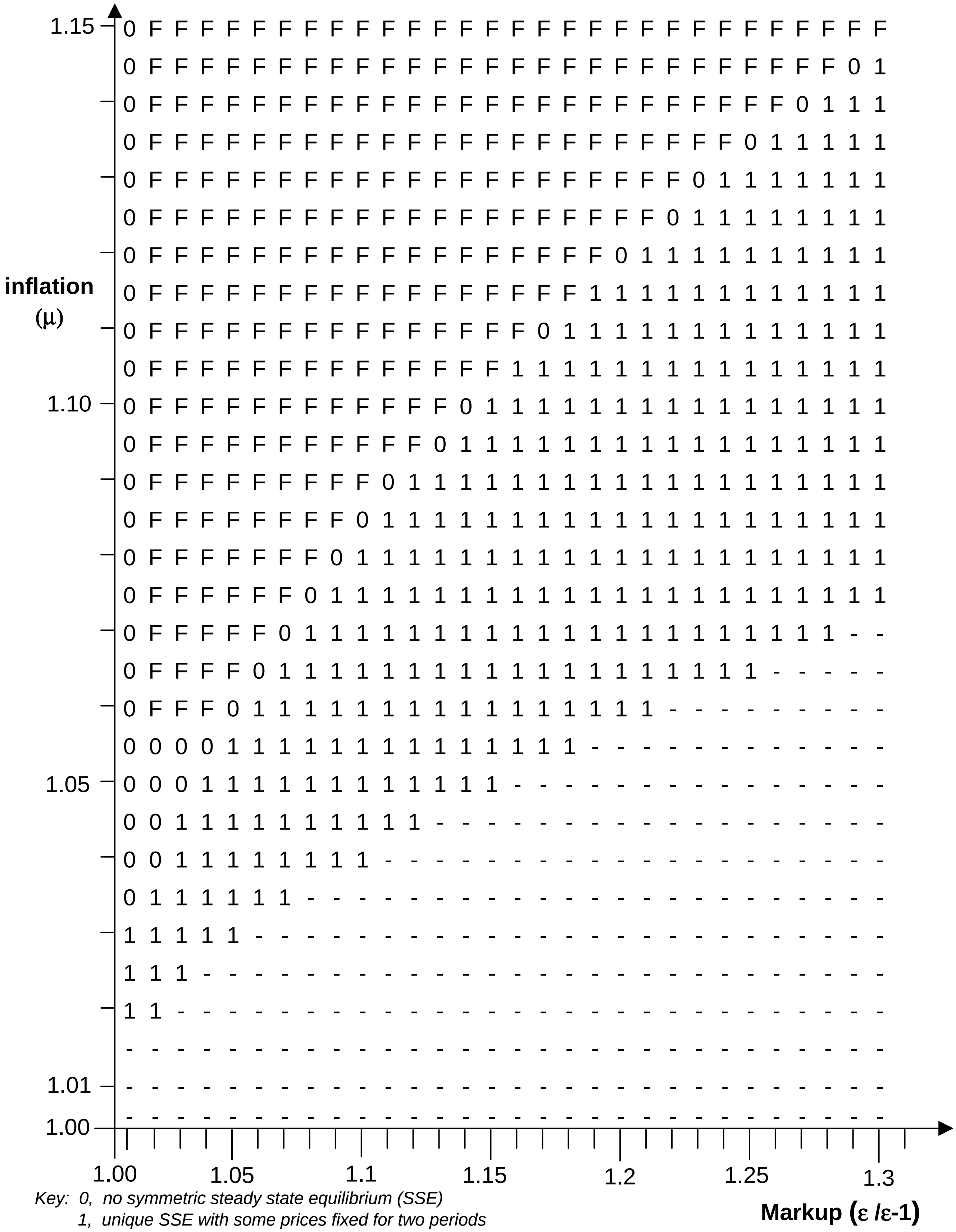


Figure 4.
Steady state equilibria
Benchmark case
(β=.975,χ=3,B=.01)



Key: 0, no symmetric steady state equilibrium (SSE)
1, unique SSE with some prices fixed for two periods
F, unique SSE with flexible prices
-, some prices fixed for more than two periods

Figure 5.
Steady state equilibria.
Concentrated distribution of fixed costs
(a=b= 10)

1.15	0	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F
	0	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	0
	0	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	0	1	1
	0	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	0	1	1	1	1
inflation	0	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	0	1	1	1	1	1	1
(μ)	0	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	1	1	1	1	1	1	1	1
	0	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	0	1	1	1	1	1	1	1	1	1
	0	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	0	1	1	1	1	1	1	1	1	1	1	1
	0	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	F	F	F	F	F	F	F	F	F	F	F	F	F	F	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1.10	0	F	F	F	F	F	F	F	F	F	F	F	F	F	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	F	F	F	F	F	F	F	F	F	F	F	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	F	F	F	F	F	F	F	F	F	F	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	F	F	F	F	F	F	F	F	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	F	F	F	F	F	F	F	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	F	F	F	F	F	F	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	F	F	F	F	F	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	F	F	F	F	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	F	F	F	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-	-	-	-
	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-	-	-	-	-	-	-	-
1.05	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-	-	-	-	-	-	-	-	-	-	-	-
	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
	0	0	1	1	1	1	1	1	1	1	1	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
	0	1	1	1	1	1	1	1	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
	1	1	1	1	1	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
	1	1	1	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
	1	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
1.01	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
1.00																														

1.00 1.05 1.10 1.15 1.20 1.25 1.30

Markup ($\varepsilon/\varepsilon_{-1}$)

Figure 6. Best-response functions
for different distributions of fixed costs
markup = 1.18, inflation = 1.06

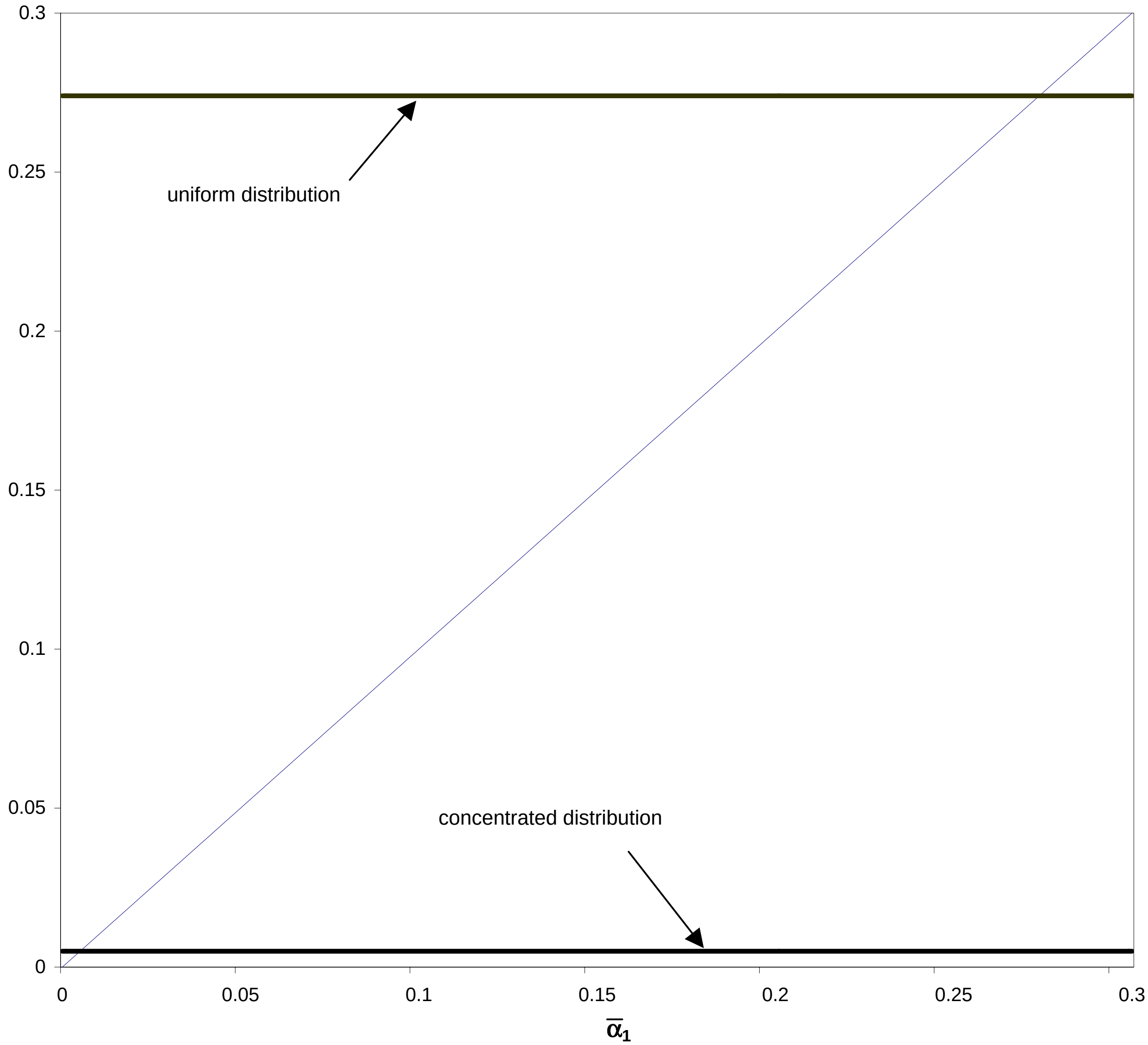
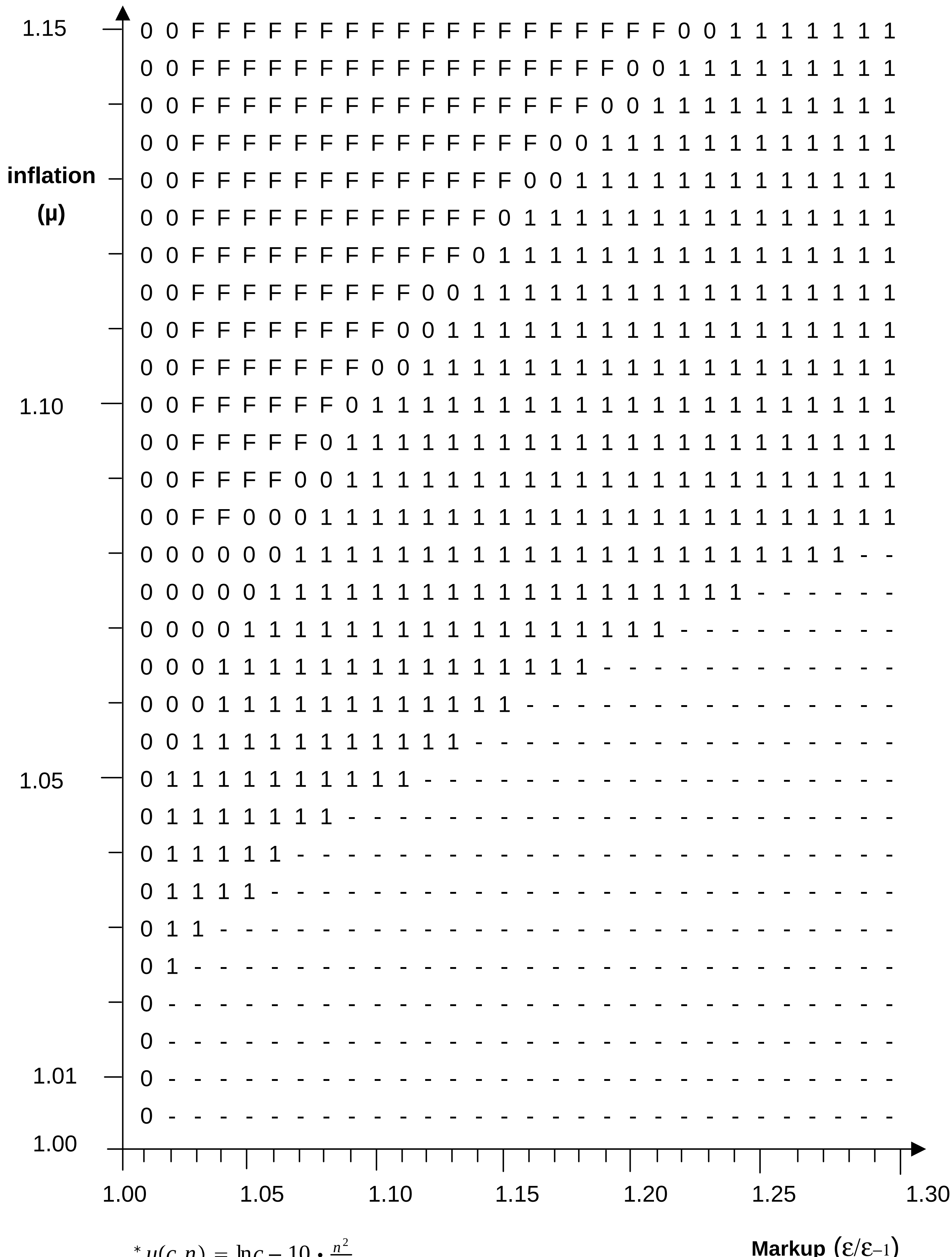


Figure 7.
Steady state equilibria.
Lower labor-supply elasticity.*



* $u(c,n) = \ln c - 10 \cdot \frac{n^2}{2}$

Figure 8.
Steady state equilibria.
Cash in advance.

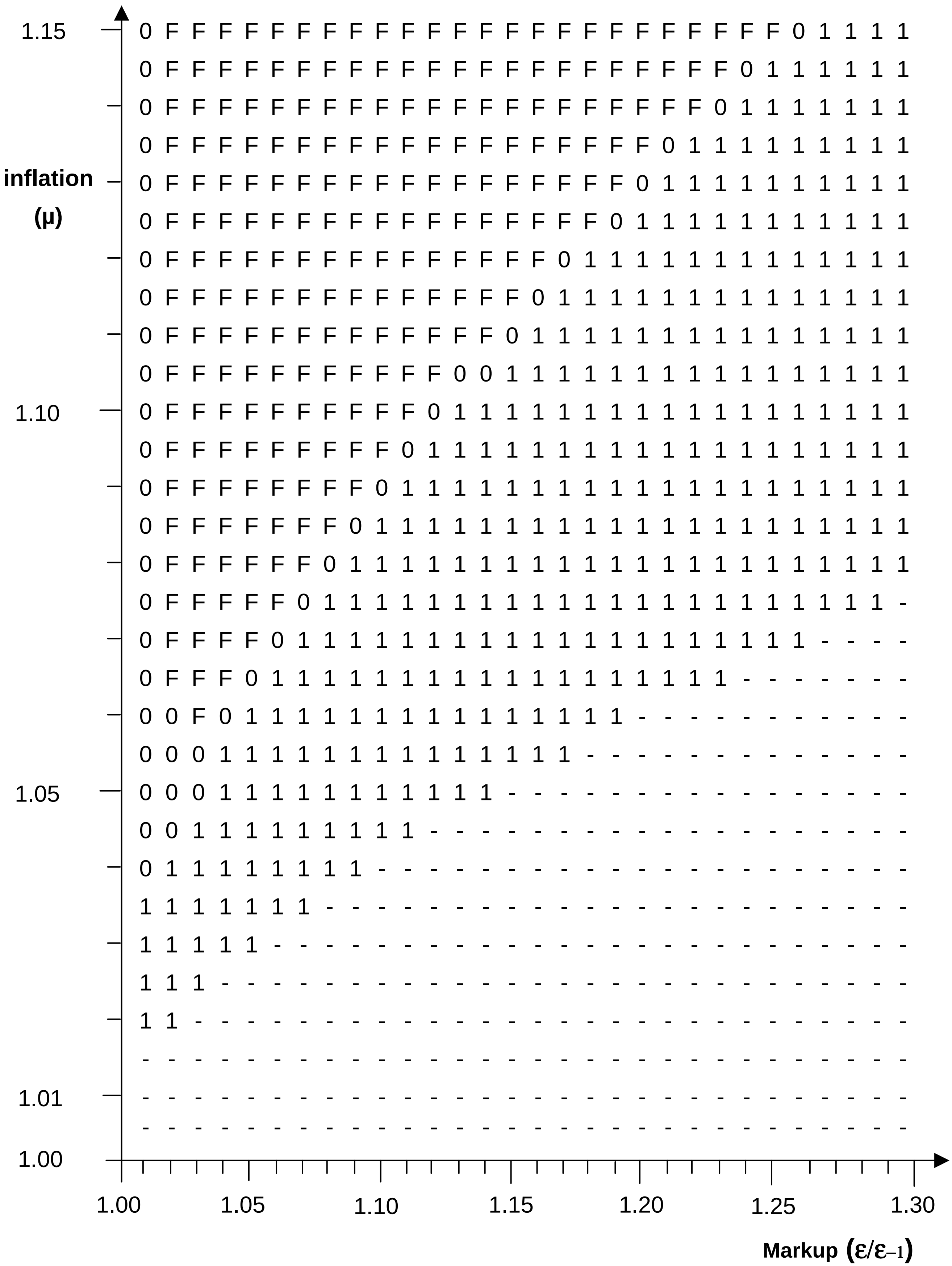


Figure 9: Non-constant demand elasticity,
steady state best response functions

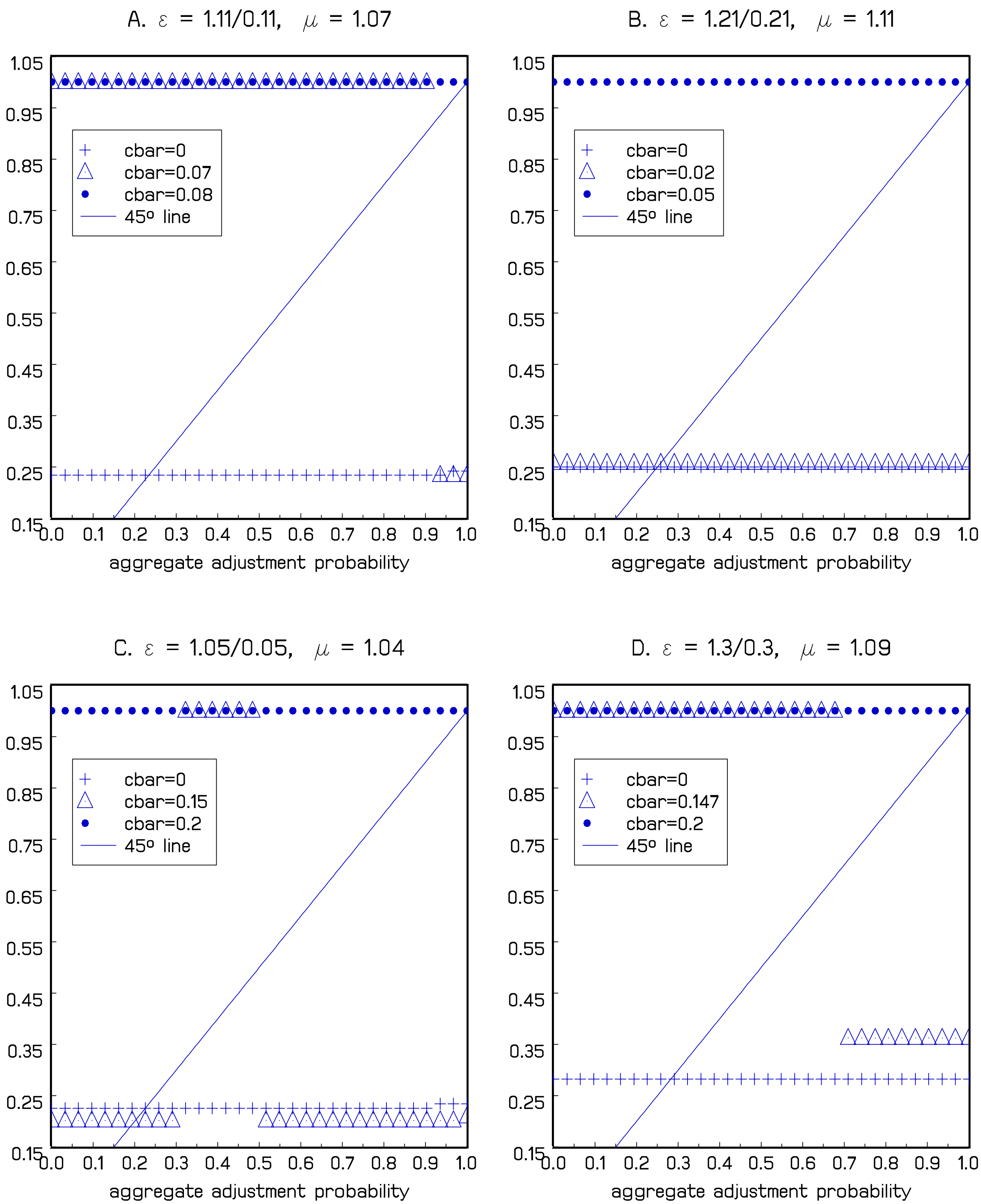


Figure 10: A one-time monetary shock.

Best response functions for α in impact period.