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Lending Relationships and Optimal Monetary Policy

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Abstract

We construct and calibrate a monetary model of corporate finance with endogenous formation of lending relationships. The equilibrium features money demands by firms that depend on their access to credit and a pecking order of financing means. We describe the mechanism through which monetary policy affects the creation of relationships and firms' incentives to use internal or external finance. We study optimal monetary policy following an unanticipated destruction of relationships under different commitment assumptions. The Ramsey solution uses forward guidance to expedite creation of new relationships by committing to raise the user cost of cash gradually above its long-run value. Absent commitment, the user cost is kept low, delaying recovery.

JEL Classification: D83, E32, E51
Keywords: Credit relationships, banks, corporate finance, optimal monetary policy.

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1 Introduction

Most businesses, especially small ones, rely on secure access to credit through stable relationships with banks. According to the 2003 Survey of Small Business Finances, 68% had access to a credit line or revolving credit arrangement – a proxy for lending relationships. As documented in Section 2, these firms hold 20% less cash relative to firms that are not in a lending relationship, thereby suggesting some degree of substituability between internal finance with cash and external finance through banking relationships. Similarly, Compustat firms with access to a credit line hold 58% less cash relative to firms without access. Insofar as monetary policy affects the user cost of cash, these observations suggest an asymmetric transmission of monetary policy to firms depending on their access to lending relationships.

Monetary policy transmission to relationship lending is especially critical in times of financial crisis as a fraction of these relationships gets destroyed due to bank failures, stricter application of loan covenants, or tighter lending standards.[1] During the Great Depression, the destruction of lending relationships explained one-eighth of the economic contraction (Cohen, Hachem, and Richardson 2016). The goal of this paper is to understand the mechanism through which monetary policy affects the creation of lending relationships and the financing of firms, and the policymaker's trade-offs in normal times and times of crisis.

We develop a general equilibrium model of lending relationships and corporate finance in the tradition of the New Monetarist approach (surveyed in Lagos, Rocheteau, and Wright 2017) and use it to study optimal monetary policy in the aftermath of a crisis. In the model economy, entrepreneurs receive idiosyncratic investment opportunities, as in Kiyotaki and Moore (2005), which can be financed with bank credit or retained earnings in liquid assets. We assume the rate of return of liquid assets, and hence the interest rate spread between liquid and illiquid assets, is controlled by the monetary authority. In the presence of search and information frictions, relationships take time to form and are costly to monitor. External finance through banks plays an essential role, even when the opportunity cost of liquidity is zero, since entrepreneurs cannot perfectly self-insure against idiosyncratic investment opportunities. The role of banks consists of issuing IOUs that are acceptable means of payment (inside money) in exchange for the illiquid IOUs of the entrepreneurs

[1] During the Great Recession, the number of small business loans contracted by a quarter from its peak (FFIEC Call Reports; Chen, Hanson, and Stein 2017), and distressed banks reneged on precommitted, formal lines of credit (Huang 2009).

with whom they have a relationship. We assume away any form of ad hoc regulation to focus squarely on the transmission mechanism that arises solely from lending relationships.

The transmission of monetary policy operates through two distinct channels. There is a liquidity channel where a fall in the rate of return of liquid assets raises the interest spread between liquid and illiquid assets and decreases holdings of liquidity for all firms. Consistent with the evidence (see e.g., Section 2), this effect is asymmetric across firms with different access to credit. Under fairly general conditions, firms in a lending relationship hold less liquid assets than unbanked firms, and this gap widens as the interest spread between liquid and illiquid assets increases. Banked firms respond more strongly to an increase in the user cost of liquid assets than unbanked firms by substituting away from internal finance into external finance.

Second, there is a novel lending channel operating through the creation of relationships. An increase in the interest rate spread between liquid and illiquid assets makes it more profitable for a firm to be in a banking relationship. Indeed, relationship lending allows firms to economize on their holdings of liquid assets, and the associated cost saving increases as liquidity becomes more expensive. Critically, because banks have some bargaining power, they can raise the revenue they collect from firms through higher interest payments or fees, which gives them incentives to create more of these relationships.

We put our model to work by investigating the economy's response to a negative credit shock described as an exogenous and unanticipated destruction of lending relationships starting from steady state. Under a policy rule that keeps the supply of liquid assets constant, the interest spread jumps up initially, thereby stimulating the creation of new relationships, before gradually declining to its initial level. In contrast, if the supply of liquid assets is perfectly elastic, aggregate liquidity increases while the rate of credit creation remains constant. In a calibrated version of the model, the policy that consists in keeping the supply of liquidity constant generates a decline in aggregate investment which is twice as large as the one obtained under a constant interest rate, but the recovery in terms of lending relationships is faster, i.e., the half time is reached about 7 months sooner.

We then turn to studying the optimal monetary policy response under different assumptions regarding the commitment power of the policymaker. If the policymaker can commit, optimal policy entails setting low spreads close to the zero lower bound at the outset of the crisis to promote internal finance by newly unbanked firms. To maintain banks' incentives to participate in the market for relationships despite low interest spreads, the policymaker uses "forward guidance" by promising high spreads in the future. If the shock is sufficiently large, the time path of spreads is hump-shaped, i.e., spreads in the medium run overshoot their long run value.

If the policymaker cannot commit and sets the interest spread period by period, then the optimal policy consists in lowering the spread to its lower bound, zero, permanently if the shock is small. If the shock is large, the optimal policy maintains banks' incentives to create lending relationships by raising spreads in the short run since it cannot commit to raise them in the future. The recovery is considerably slower than under commitment; e.g., the half life of the transition path to steady state following a 60% contraction absent commitment is more than 27 months in our calibrated example, compared with 20 months under the Ramsey policy. This produces a welfare loss from lack of commitment ranging from 0.90% to 0.98% of consumption, depending on the size of the shock. Our model is not restricted to the study of banking crises and we conclude by characterizing the optimal response to a policy shock that consists of a partial lock-down of the economy with the COVID-19 crisis in mind.

Literature

There are four main approaches to the role of lending relationships: insurance in the presence of uncertain investment projects (Berlin and Mester 1999), monitoring in the presence of agency problems (Holstrom and Tirole 1997), screening with hidden types (Agarwal and Hauswald 2010), and dynamic learning under adverse selection (Sharpe 1990, Hachem 2011).[2] We adopt the insurance approach as it is central to the monetary policy tradeoff we are focusing on. Our assumption on the costs of external finance is related to Diamond (1984).

There is a small literature on monetary policy and relationship lending, e.g., Hachem (2011) and Bolton, Freixas, Gambacorta, and Mistrulli (2016).[3] Our model differs from that literature by emphasizing money demands by firms and their choice between internal and external finance, endogenizing the creation of relationships through a frictional matching technology, and assuming banks have bargaining power. Our description of the credit market with search frictions is analogous to den Haan, Ramey, and Watson (2003), Wasmer and Weil (2004), Petrosky-Nadeau and Wasmer

[2] See Elyasiani and Goldberg (2004) and references therein for a survey of the corporate finance literature on relationship lending.

[3] Bolton and Freixas (2006) is a related paper that focuses on monetary policy and transaction lending. Boualam (2017) models relationship lending with directed search and agency costs but does not have an endogenous demand for liquid assets or monetary policy.

(2017), and models of OTC dealer markets by Duffie, Garleanu, and Pedersen (2005) and Lagos and Rocheteau (2009). Drechsler, Savov, and Schnabl (2017) also assume banks have market power but focus on the deposits market in a static model with cash and deposits in the utility function. In addition, we characterize the optimal monetary policy under different assumptions on the policymaker's commitment.

Our model is a corporate finance version of Lagos and Wright (2005) and its competitive version by Rocheteau and Wright (2005). The closest papers are Rocheteau, Wright, and Zhang (2018) which studies transaction lenders when firms are subject to pledgeability constraints. Imhof, Monnet, and Zhang (2018) extends the model by introducing limited commitment by banks and risky loans.[4] Our approach to the coexistence of money (liquid assets) and bank credit is related to Sanches and Williamson (2010) who assume that cash is subject to theft. Our formalization of banks is similar to the one in Gu, Mattesini, Monnet, and Wright (2016) and references therein. Models of money and credit with long-term relationships include Corbae and Ritter (2004) with indivisible money and Rocheteau and Nosal (2017, Ch. 8) with divisible money. Our description of a crisis is analogous to the one in Weill (2007) and Lagos, Rocheteau, and Weill (2011).

Our recursive formulation of the Ramsey problem is related to Chang (1998) and Aruoba and Chugh (2010) in the context of the Lagos-Wright model. Our approach to the policy problem without commitment is similar to Klein, Krusell, and Rios-Rull (2008) and Martin (2011, 2013) in a New Monetarist model where the government finances the provision of a public good with money, nominal bonds, and distortionary taxes. Unlike the usual perturbation method applying to the steady state, we devise an algorithm based on contraction mappings to compute the entire transitional dynamics.

2 Empirical support

Here we provide some empirical observations on money demands by firms contingent on their access to a lending relationship. We also document the link between the user cost of liquid assets and the profitability of small bank loans that we associate to relationship lending to small businesses.

[4] Here a lending relationship is a commitment by the bank to provide firms with conditional access to credit. While we do not have a credit limit under limited commitment here, as in e.g. Kehoe and Levine (1993) or Alvarez and Jermann (2000), there is a limit on banks' willingness to lend due to enforcement costs. See Raveendranathan (2020) for a model of revolving credit lines where a credit contract specifies an interest rate and credit limit.

Observation #1: Firms' demand for liquid assets. We identify firms' money demand in the data by using the cash-assets ratio for small businesses from the 2003 Survey of Small Business Finances (SSBF). Cash is defined as "any immediately negotiable medium of exchange," which includes certificates of deposit (CDs), checks, demand deposits, money orders, and bank drafts. The user cost of cash is based on the Divisia monetary aggregate, MSI-ALL, developed by Barnett (1980).[5]

We estimate money demand by banked and unbanked firms, controlling for various sources of firm heterogeneity, by running the following regression:

$$\log(m_{i,t}) = \beta_b D_{i,t} + e_u(1 - D_{i,t})s_t + e_b D_{i,t} s_t + X_{i,t} \cdot \gamma + \Upsilon_t + \epsilon_{i,t}, \tag{1}$$

where $m_{i,t}$ is cash-assets of firm i in year t, s_t is the user cost of cash in year t (common across firms), $D_{i,t} \in \{0, 1\}$ is an indicator that equals one if firm i has access to a line of credit in year t, X_{it} is a vector of controls, including a constant, capturing firm i's attributes and financial characteristics in year t, Υ_t is a vector of time fixed effects, and $\epsilon_{i,t}$ is an error term assumed to be independent across firms but not necessarily across time. Firms in this sample completed the survey on different dates from 2003 to 2005. Hence we match each SSBF sample with the user cost on the completed date. We include a standard set of controls that capture different firm attributes and financial characteristics and estimate (1) using robust standard errors to allow for correlation in the error term within firms and across time.[6]

For robustness, we also estimate a similar regression as (1) using Compustat data which is not a one-time survey like the SSBF. In the Compustat data, $m_{i,t}$ is cash-sales and $D_{i,t}$ is obtained by merging firm level cash-sales with the dataset in Sufi (2009) which contains information on whether a firm has access to a line of credit from annual SEC 10-K filings from 1996 to 2003. Controls in the Compustat regression include age, size, and financial characteristics such as cash flow, net worth, market to book, and book leverage.

[5] We use the monetary aggregate at the broadest level of aggregation as our measure of liquid assets, which is constructed over all assets reported in the Federal Reserve Board's H.6 statistical release, i.e. the components of M2 plus institutional money market mutual funds. The user cost is the spread between the own rate of return from holding the portfolio of MSI-ALL and a benchmark rate that equals 100 basis points plus the maximum of the interest rate on short-term money market rates and the largest interest rate out of the components of MSI-ALL.

[6] Controls in the SSBF regressions include firm attributes and financial characteristics. Firm attribute variables include firm industry, urban or rural location, corporation type, how the firm is acquired, as well as productivity related variables like the return on asset, profit growth, owner's age, owner's years of experience, and level of education. Financial characteristic variables include the HHI banking concentration index in the owner's area, financial status variables like their credit score, whether or not the firm or owner has filed bankruptcy, the length of relationship with the credit line lenders, and variables related to financial discrimination like race, gender, and age of the owners. All variables include higher order terms and interactions to capture potential non-linearities.

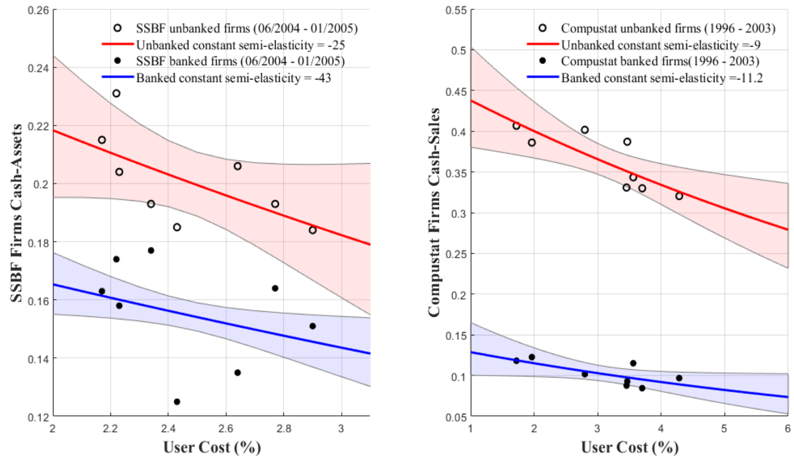

Figure 1: Demand for liquid assets by banked and unbanked firms from SSBF (left) and Compustat (right)

From the SSBF data, we obtain $\exp(-\beta_b) = 1.25$, significant at the 1% level. Hence small businesses in the SSBF who are not in a lending relationship hold 25% more cash as unbanked firms, controlling for monetary policy and various firm characteristics. This compares with an estimate of 2.22 with Compustat data, significant at the 1% level. The estimate is higher among Compustat firms, which may be due to selection bias: e.g., cash-rich firms do not need bank credit in the first place. We control for this bias in the SSBF regression by only focusing on firms who are actively looking for bank credit (this information is not available in the Compustat data). Figure 1 summarizes firms' money demand separately for SSBF (left) and Compustat (right) firms with and without a line of credit.

The user cost semi-elasticities for the demand for liquid assets by unbanked and banked firms are $e_u = \partial \log(m^u)/\partial s = -25$ and $e_b = \partial \log(m^b)/\partial s = -43$ from the SSBF data. Similarly, banked firms' cash demand in Compustat is more sensitive to interest spreads than unbanked firms; the Compustat semi-elasticities are $e_u = -9$ and $e_b = -11$.

Observation #2: Profitability of small loans and the user cost of liquid assets. To explore the relation between banks' profitability of loans to small businesses and the user cost of cash, we use bank level data from the FFIEC's Call Reports. We define small business loans as loans less than \$1 million. We focus on banks making more than half of their commercial and industrial (C&I) loans less than \$1 million. The Net Interest Margin (NIM) for small business loans is measured as:[7]

$$\text{NIMsb} = \underbrace{\frac{\text{interest \& fee income on C\&I loans} < \$1\text{m}}{\text{C\&I loans} < \$1\text{m}}}_{\text{loan rate on small business loans}} - \underbrace{\frac{\text{total interest expense}}{\text{total assets}}}_{\text{funding cost}}. \tag{2}$$

The first term measures banks' revenue from interest and fees while the second term is banks' cost of funds. The left panel in Figure 2 shows the user cost (black line) and loan rate on small business loans (red line) is highly correlated over time; the correlation coefficient is 0.7. The right panel shows the relationship between the user cost and small business NIM (blue line) is also positive; the correlation coefficient is 0.25 although higher before the financial crisis (0.64).

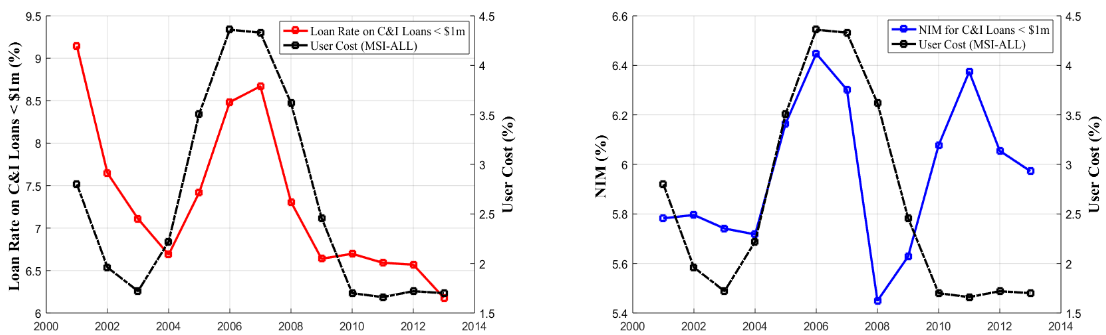

Figure 2: Loan rate (left), net interest margin (right), and user cost of liquid assets

We estimate a linear regression of NIM for small business loans against the user cost and macroeconomic controls:

$$\log(\text{NIMsb}_t) = \beta_0 + \beta_1 \log(s_t) + \beta_2 \log(\text{NIMsb}_{t+1}) + X_t \cdot \gamma + \epsilon_t, \tag{3}$$

where $NIMsb_t$ is NIM for small business loans in year t, s_t is the user cost for MSI-ALL, and X_t is a vector of controls for macroeconomic and credit conditions, such as GDP growth and overall bank lending. Our econometric specification follows the model's prediction that NIMsb_{t+1} captures the dependence of current profitability on future profits. Estimating (3) using U.S. Call Report data from 2001 to 2013 with robust standard errors, we obtain an elasticity of small business NIM with respect to the user cost of 0.06, significant at the 10% level.[8]

[7] We describe in detail our construction and report additional results in the Supplementary Data Appendix.

[8] These findings are consistent with evidence from Claessens, Coleman, and Donnelly (2017), Borio, Gambacorta, and Hoffmann (2018), Berry, Ionescu, Kurtzman, and Zarutskie (2019), and Grochulski, Schwam, and Zhang (2018). If we run the same regression as (3) using NIM for all banks' assets we obtain a negative elasticity with respect to the spread, though this is not statistically significant. The tenuous relationship between NIM for all banks' assets and interest rates is discussed by Ennis, Fessenden, and Walter (2016). Balloch and Koby (2020) find evidence from Japan on the adverse effects of low interest rates on bank profitability and loan supply. See also our discussion in the Supplementary Data Appendix.

3 Environment

Time is indexed by $t \in \mathbb{N}_0$. Each period is divided in three stages. In the first stage, a competitive market for capital goods opens and investment opportunities arise. The second stage is a frictional market where long-term lending relationships are formed. The last stage is a frictionless centralized market where agents trade assets and consumption goods and settle debts. Figure 3 summarizes the timing of a representative period.

There are two goods: a capital good k storable across stages but not across periods and a consumption good c taken as the numéraire. There are three types of agents: *entrepreneurs* who need capital, *suppliers* who can produce capital, and *banks* who can finance the acquisition of capital as explained below. The population of entrepreneurs is normalized to one. Given CRS for the production of capital goods (see below), the population size of suppliers is immaterial. The population of active banks is endogenous and will be determined through free entry. All agents have linear preferences, $c - h$, where c is consumption of numéraire and h is labor. They discount across periods according to $\beta = 1/(1+\rho)$, where $\rho > 0$.

In stage 1, entrepreneurs have probabilistic access to a technology that transforms k units of capital goods into $y(k)$ units of numéraire in stage 3. We assume $y(k)$ is continuously differentiable with $y' > 0$, $y'' < 0$, $y'(0) = +\infty$, and $y'(+\infty) = 0$. Production/investment opportunities are iid across time and entrepreneurs and they occur with probability λ^u for unbanked entrepreneurs and λ^b for banked entrepreneurs. We assume $\lambda^b \geq \lambda^u$ to capture the role of banks in generating information regarding investment opportunities while monitoring the activity of the entrepreneurs they are matched with.[9] Capital k is produced by suppliers in stage 1 with a linear technology, $k = h$. Social efficiency dictates $k = k^*$ where $y'(k^*) = 1$. Agents can also produce c using their labor in stage 3 with a linear technology, $c = h$.

Entrepreneurs lack commitment, have private trading histories, and do not interact repeatedly with the same suppliers. As a result, suppliers do not accept IOUs issued by entrepreneurs who have no consequences to fear from reneging.[10] In contrast, banks have access to a commitment technology that allows them to issue liabilities that are repaid in the last stage. Banks also have

[9] Herrera and Minetti (2007) and Cosci, Meliciani, and Sabato (2016) find banking relationships raise the probability that a firm innovates. This is consistent with the notion that lending relationships enhance the flow of information to the bank (e.g., Petersen and Rajan 1994, Berger and Udell 1995).

[10] Alternatively, we could allow for trade credit in a fraction of matches as in Rocheteau, Wright, and Zhang (2018).

the technology to enforce the repayment of entrepreneurs' IOUs.[11] These technologies are operated at a cost $\psi(\text{L})$, where L is both the liabilities (in terms of the numéraire) issued by the bank to be repaid in stage 3 and the principal on the entrepreneur's loan. This cost is increasing and convex, i.e., $\psi'(\text{L}) > 0$, $\psi''(\text{L}) > 0$, $\psi(0) = \psi'(0) = 0$. It includes the costs to issue liabilities that are easily recognizable and non-counterfeitable, costs associated with the commitment to repay L, and costs to monitor the entrepreneurs' loans backing these liabilities.

To be eligible for external finance, an entrepreneur must form a lending relationship with a bank. Each bank manages at most one relationship.[12] These relationships are formed in stage 2. At the beginning of stage 2, banks without a lending relationship decide whether to participate in the credit market at a disutility cost, $\zeta > 0$. There is then a bilateral matching process between unbanked entrepreneurs and unmatched banks. The number of new relationships formed in stage 2 of period t is $\alpha_t = \alpha(\theta_t)$, where θ_t is the ratio of unmatched banks to unbanked entrepreneurs, defined as credit market tightness. We assume $\alpha(\theta)$ is increasing and concave, $\alpha(0) = 0$, $\alpha'(0) = 1$, $\alpha(\infty) = 1$, and $\alpha'(\infty) = 0$. Since matches are formed at random, the probability an entrepreneur matches with a bank is α_t, and the probability a bank matches with an entrepreneur is $\alpha_t^b = \alpha(\theta_t)/\theta_t$. We denote the elasticity of the matching function $\epsilon(\theta) \equiv \alpha'(\theta)\theta/\alpha(\theta)$. A match existing for more than one period is terminated at the end of the second stage with probability $\delta \in (0, 1)$. Newly formed matches are not subject to the risk of termination.

In addition to banks' short-term liabilities, there are risk-free assets which are storable across periods and promise a real rate of return from period t to $t+1$ equal to r_{t+1}. We assume r_{t+1} is set by the policymaker; e.g., the policymaker determines the supply of money or government bonds.[13] If liquid assets take the form of fiat money, then r_{t+1} corresponds to the opposite of the inflation rate and is implemented through changes in the money growth rate.

Liquid assets held by entrepreneurs are subject to embezzlement or theft at the beginning of stage 1, before investment opportunities arise, with probability $1-\nu$. For instance, an entrepreneur must hire a manager or worker who can run away with the entrepreneur's liquid assets in the early morning with probability $1 - \nu$ with no consequences.[14] If theft happens, the entrepreneur has

[11] We endogenize bank commitment through reputation in the appendix of Rocheteau, Wright, and Zhang (2018). The possibility of insolvent banks in a version of our model is studied by Imhof, Monnet, and Zhang (2018).

[12] This assumption is analogous to the Pissarides (2000) one-firm-one-job assumption. One can think of actual banks as a large collection of such relationships.

[13] While we ignore the potential fiscal implications of adjusting interest rates, one can think of those as financed with lump-sum transfers or taxes.

[14] The role theft to explain the co-essentiality of money and credit was first emphasized in He, Huang, and Wright

no assets left to finance potential investment opportunities. With complementary probability ν, embezzlement does not take place and the entrepreneur can use his liquid asset to finance investment opportunities.

4 Liquidity and lending relationships

We study equilibria where investment opportunities are financed with bank loans and liquid assets accumulated from retained earnings.

4.1 Value functions

Notations for value functions of entrepreneurs and banks in different states and different stages are summarized in Figure 3. We characterize these value functions from stage 3 and move backwards to stage 1.

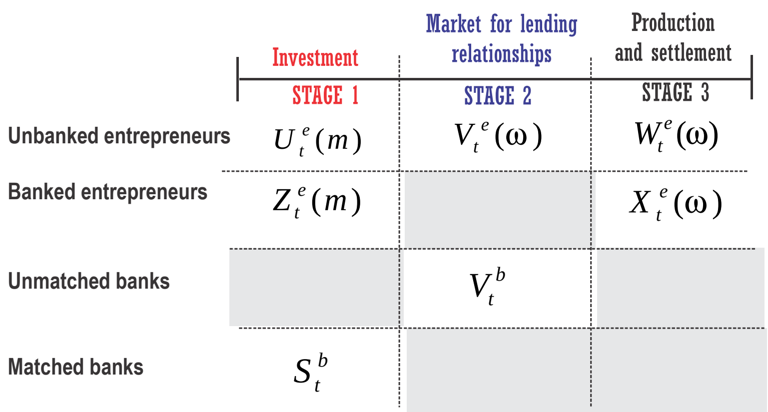

Figure 3: Timing of a representative period and value functions

STAGE 3 (Settlement and portfolio choices). The lifetime expected utility of an unbanked entrepreneur with wealth ω (expressed in terms of numéraire) in the last stage of period t is

$$W_t^e(\omega) = \max_{c_t, h_t, m_{t+1} \geq 0} \left\{ c_t - h_t + \beta U_{t+1}^e(m_{t+1}) \right\} \quad \text{s.t.} \quad m_{t+1} = (1 + r_{t+1})(\omega + h_t - c_t),$$

where $U_t^e(m)$ is the value function of an unbanked entrepreneur at the beginning (stage 1) of period t with liquid wealth m. The entrepreneur saves $\omega + h_t - c_t$ from his current wealth and income in the form of liquid assets. The rate of return on liquid assets is r_{t+1}, hence holdings in period

(2005) and Sanches and Williamson (2010). According to the Hiscox Embezzlement Study (2018), corporate theft includes billing fraud (18%), theft of cash on hand (15%), check and payment tampering (10%), payroll theft (10%), skimming (9%), and cash larceny (9%).

$t+1$ are $m_{t+1} = (1+r_{t+1})(\omega + h_t - c_t)$. Substituting $c_t - h_t$ from the budget identity into the objective, the Bellman equation becomes

$$W_t^e(\omega) = \omega + \max_{m_{t+1}\geq 0}\left\{-\frac{m_{t+1}}{1+r_{t+1}} + \beta U_{t+1}^e(m_{t+1})\right\}. \tag{4}$$

As is standard in models with risk-neutral agents, value functions are linear in wealth and the choice of m_{t+1} is independent of ω. By a similar reasoning, the lifetime utility of a banked entrepreneur with wealth ω in the last stage of period t, $X_t^e(\omega)$, solves

$$X_t^e(\omega) = \omega - \frac{m_{t+1}^b}{1+r_{t+1}} + \beta Z_{t+1}^e(m_{t+1}^b), \tag{5}$$

where $Z_t^e(m)$ is the value of a banked entrepreneur at the beginning of period t with m units of liquid assets and m_{t+1}^b is the amount of liquid assets that the banked entrepreneur must hold as specified by the lending relationship contract.

STAGE 2 (Market for lending relationships). The lifetime expected utility of an unbanked entrepreneur at the beginning of the second stage solves:

$$V_t^e(\omega) = \alpha_t X_t^e(\omega) + (1-\alpha_t)W_t^e(\omega) = \omega + \alpha_t X_t^e(0) + (1-\alpha_t)W_t^e(0). \tag{6}$$

With probability α_t, the unmatched entrepreneur enters a lending relationship and, with probability $1-\alpha_t$, he proceeds to the last stage unmatched. From the right side of (6), $V_t^e(\omega)$ is linear in ω.

The lifetime discounted profits of a bank entering at time t, V_t^b, solve

$$V_t^b = -\zeta + \alpha_t^b \beta \mathcal{S}_{t+1}^b + \left(1-\alpha_t^b\right)\beta \max\left\{V_{t+1}^b, 0\right\}. \tag{7}$$

From (7), an unmatched bank incurs a cost ζ at the start of the second stage to participate in the credit market; there, the bank is matched with an entrepreneur with probability $\alpha_t^b = \alpha(\theta_t)/\theta_t$ and remains unmatched with probability $1-\alpha_t^b$. The discounted sum of the profits from a lending relationship is $\mathcal{S}_{t+1}^b$.

STAGE 1 (Investment opportunities). In the first stage, suppliers choose the amount of k to produce at a linear cost taking its price in terms of numéraire, q_t, as given. Formally, they solve $\max_{k\geq 0}\{-k + q_t k\}$. If the capital market is active, $q_t = 1$. The lifetime utility of an unbanked entrepreneur at the beginning of period t is

$$U_t^e(m_t) = \mathbb{E}\left[V_t^e(\omega_t)\right] \quad \text{s.t.} \quad \omega_t = \mathbb{I}_t\left\{m_t + \chi_t^u \max_{k_t\leq m_t}\left[y(k_t) - k_t\right]\right\}, \tag{8}$$

where $\mathbb{I}_t$ is Bernoulli variable equal to one if liquid assets are not embezzled, which occurs with probability ν, and χ_t^u is an independent Bernoulli variable equal to one with probability λ^u if the entrepreneur receives an investment opportunity. If no theft occurred, then the entrepreneur's wealth when entering the second stage, ω_t, consists of his initial wealth, m_t, and profits from the investment opportunity if $\chi_t^u = 1$. To maximize profits, the entrepreneur chooses k_t subject to the liquidity constraint $k_t \leq m_t$. By the linearity of $V_t^e(\omega_t)$, (8) reduces to:

$$U_t^e(m_t) = \lambda^u \nu \max_{k_t \leq m_t} [y(k_t) - k_t] + \nu m_t + V_t^e(0). \tag{9}$$

The lifetime expected utility of a banked entrepreneur with m liquid assets at the beginning of stage 1 solves

$$Z_t^e(m) = \mathbb{E}\left[\delta W_t^e(\omega_t) + (1-\delta) X_t^e(\omega_t)\right], \tag{10}$$
$$\text{s.t.} \quad \omega_t = \mathbb{I}_t m + \chi_t^b \left\{ \mathbb{I}_t \left[y(k_t^b) - k_t^b \right] + (1 - \mathbb{I}_t) \left[y(\hat{k}_t^b) - \hat{k}_t^b \right] \right\} - \phi_t,$$

where k_t^b is the investment level when entrepreneurs are not subject to theft, $\hat{k}_t^b$ is the investment level when liquid assets are subject to embezzlement, and ϕ_t is an intermediation fee due in stage 3. The indicator variable, χ_t^b, equals one if the banked entrepreneur receives an investment opportunity with probability λ^b. The quantities, $(k_t^b, \hat{k}_t^b, \phi_t)$, are determined as part of an optimal contract. Using the linearity of $W_t^e(\omega_t)$ and $X_t^e(\omega_t)$,

$$Z_t^e(m) = \mathbb{E}[\omega_t] + \delta W_t^e(0) + (1-\delta) X_t^e(0), \tag{11}$$

where the entrepreneur's expected wealth at the end of a period is

$$\mathbb{E}[\omega_t] = \nu m - \phi_t + \lambda^b \nu \left[y(k_t^b) - k_t^b \right] + \lambda^b (1-\nu) \left[y(\hat{k}_t^b) - \hat{k}_t^b \right],$$

From (11), the lending relationship is destroyed with probability δ, in which case the entrepreneur's value in the last stage is W_t^e. Otherwise, the continuation value is X_t^e.

Finally, the discounted sum of bank profits from a lending relationship at the start of period t, $\mathcal{S}_t^b$, solves

$$\mathcal{S}_t^b = \phi_t - \lambda^b \nu \psi \left(k_t^b - m_t^b \right) - \lambda^b (1-\nu) \psi \left(\hat{k}_t^b \right) + \beta (1-\delta) S_{t+1}^b, \tag{12}$$

where m_t^b is the holdings of liquid assets of the entrepreneur the bank is matched with to be used as down payment for a loan. The second and third terms on the right side are the cost of the loans $\mathrm{L}_t = k_t^b - m_t^b$ and $\hat{\mathrm{L}}_t = \hat{k}_t^b$, respectively.

4.2 Optimal liquidity of unbanked entrepreneurs

We now determine the optimal holdings of liquid assets by unbanked entrepreneurs. Substituting $U^e_t(m_t)$ from (9) into (4), an unbanked entrepreneur's choice of liquid assets is a solution to:

$$\pi^u_t = \pi^u(s_t) \equiv \nu \max_{m_t \geq 0} \left\{ -s_t m_t + \lambda^u \max_{k_t \leq m_t} \left[y(k_t) - k_t \right] \right\}, \tag{13}$$

where the interest rate spread between liquid and illiquid asset is

$$s_t \equiv \frac{R^i - (1+r_t)}{1+r_t}, \tag{14}$$

and $R^i \equiv (1+\rho)/\nu$ is the gross rate of return of an illiquid asset that is subject to theft with the same probability as a liquid asset, $1-\nu$.[15] If the liquid asset does not bear interest (e.g., cash), then s is the nominal rate on an illiquid bond and its lower bound is zero.[16] The FOC associated with (13) is

$$s_t = \lambda^u \left[y'(m^u_t) - 1 \right], \tag{15}$$

where m^u_t denotes the demand for liquid assets by unbanked entrepreneurs. The term on the left side is the cost of holding liquid assets whereas the right side is the expected marginal benefit from holding an additional unit of the liquid asset, which is the probability of an investment opportunity times the marginal profits from an additional unit of capital. The optimal liquid wealth of an unbanked entrepreneur decreases with s_t but increases with λ^u.

4.3 Optimal lending relationship contract

The lending relationship contract negotiated in stage 2 of period $t-1$ between newly matched entrepreneurs and banks is a list, $\{\phi_t, \hat{k}^b_{t+\tau}, k^b_{t+\tau}, m^b_{t+\tau}\}_{\tau=0}^{\infty}$, where ϕ_t is the fee to the bank, $\hat{k}^b_{t+\tau}$ is investment if the entrepreneur is victim of embezzlement and has no liquid asset, $k^b_{t+\tau}$ is investment if the entrepreneur is not victim of theft, and $m^b_{t+\tau}$ is the amount of liquid wealth to be used as down payment on loans. So, the loan size is $\hat{\mathrm{L}}_{t+\tau} = \hat{k}^b_{t+\tau}$ if the entrepreneur cannot access his liquid assets and $\mathrm{L}_{t+\tau} = k^b_{t+\tau} - m^b_{t+\tau}$ otherwise.

The entrepreneur's surplus from being in a lending relationship in the third stage of $t-1$ is defined as $\mathcal{S}^e_t = \left[X^e_{t-1}(0) - W^e_{t-1}(0) \right] / \beta$. The bank's surplus is $\mathcal{S}^b_t$. The terms of the lending

[15] Alternatively, we could express the cost of holding liquidity in terms of a spread between liquid and illiquid assets that are not subject to theft. In that case, the lower bound for s would be negative and equal to the opposite of the probability of theft. The rest of the analysis would be unaffected.

[16] This expression for the cost of holding liquid assets as a spread is consistent with the construction in Barnett (1980).

relationship contract are chosen to maximize the generalized Nash product, $\left(\mathcal{S}_t^b\right)^\eta \left(\mathcal{S}_t^e\right)^{1-\eta}$, where η is the bargaining power of banks. As is standard in bargaining problems with transferable utilities, $\{\hat{k}_{t+\tau}^b, k_{t+\tau}^b, m_{t+\tau}^b\}_{\tau=0}^\infty$ is chosen to maximize the total surplus of a lending relationship, $\mathcal{S}_t = \mathcal{S}_t^e + \mathcal{S}_t^b$, while $\{\phi_t\}$ splits the surplus according to each party's bargaining power. In the proof of Proposition 1, we show the total surplus of a relationship solves:

$$\begin{aligned}
\mathcal{S}_t &= \lambda^b(1-\nu)\left[y(\hat{k}_t^b) - \hat{k}_t^b - \psi(\hat{k}_t^b)\right] + \lambda^b\nu\left[y(k_t^b) - k_t^b - \psi(k_t^b - m_t^b)\right] \\
&\quad -\nu s_t m_t^b - \underline{\pi}_t + (1-\delta)\beta\mathcal{S}_{t+1},
\end{aligned} \tag{16}$$

where

$$\underline{\pi}_t = \pi_t^u + V_t^e(0) - W_t^e(0) \tag{17}$$

represents the opportunity cost of being in a lending relationship and is composed of the expected profits of an unbanked entrepreneur (net of the cost of holding liquid wealth), π_t^u, augmented with the value of the matching opportunities in the credit market, $V_t^e(0) - W_t^e(0)$. The first term on the right side of (16) is the expected profits from externally financed investment opportunities. The second term corresponds to the profits of an investment opportunity financed both internally with m_t^b units of liquid wealth and externally with a loan of size $\mathrm{L}_t = k_t^b - m_t^b$. The third term is the entrepreneur's cost of holding m_t^b units of liquid wealth.

Proposition 1 ***(Optimal lending contract with internal finance.)*** *The terms of the optimal lending relationship contract solve*

$$\begin{aligned}
\psi'\left(\hat{k}_t^b\right) &= y'(\hat{k}_t^b) - 1 && (18)\\
\psi'\left(k_t^b - m_t^b\right) &= y'(k_t^b) - 1 \leq \frac{s_t}{\lambda^b}, \quad \text{"="} \textit{ if } m_t^b > 0, \quad \forall t. && (19)
\end{aligned}$$

The intermediation fee is equal to

$$\phi_t = \lambda^b\left[(1-\nu)\psi(\hat{k}_t^b) + \nu\psi(k_t^b - m_t^b)\right] + \eta\left[\pi^b(s_t) - \pi^u(s_t)\right] - (1-\eta)\zeta\theta_t, \tag{20}$$

where the joint expected profits net of the cost of holding assets are

$$\begin{aligned}
\pi^b(s_t) &= \lambda^b(1-\nu)\left[y(\hat{k}_t^b) - \hat{k}_t^b - \psi(\hat{k}_t^b)\right] \\
&\quad +\nu \max_{k^b, m^b \geq 0}\left\{\lambda^b\left[y(k^b) - k^b - \psi(k^b - m^b)\right] - s_t m^b\right\}.
\end{aligned} \tag{21}$$

The optimal contract is consistent with a pecking order where firms use internal finance first to fund investment projects and resort to external finance last. Indeed, conditional on an entrepreneur holding m^b units of liquid assets, the optimal loan contract solves:

$$\varpi(m^b) = \max_{k^b, \text{L}} \left\{ y(k^b) - k^b - \psi(\text{L}) \right\} \quad \text{s.t.} \quad k^b \leq \text{L} + m^b. \tag{22}$$

If $m^b \geq k^*$, the entrepreneur finances all of the project internally, $k^b = k^*$ and $\text{L}= 0$. If $m^b < k^*$, the entrepreneur finances m^b units of capital internally and L units externally where L is chosen to equalize the net marginal return of capital, $y'(k_t^b) - 1$, and the marginal cost of external finance, $\psi'(\text{L}_t)$. Given $\varpi(m^b)$, the optimal holdings of liquid assets of a banked entrepreneur solves:

$$m^b = \arg\max_{m^b \geq 0} \left\{ \lambda^b \varpi(m^b) - s m^b \right\}. \tag{23}$$

Using that $\varpi'(m^b) = \psi'(\text{L})$, the first-order condition for the optimal holdings of liquid assets satisfies (19). The inequality in (19) states that the marginal gain from financing investment internally, $\lambda^b \psi'\left(k_t^b - m_t^b\right)$, cannot be greater than the opportunity cost of holding liquid assets, s_t.

The intermediation fee in (20) consists of the average cost of monitoring loans, a fraction η of the entrepreneurs' profits from being in a lending relationship, net of a fraction $1-\eta$ of the banks' entry costs. It depends on s_t through the term $\Delta\pi(s_t) \equiv \pi^b(s_t) - \pi^u(s_t)$ where $\partial\Delta\pi(s_t)/\partial s_t = \nu\left(m_t^u - m_t^b\right)$. The difference in liquid wealth between unbanked and banked entrepreneurs provides a channel through which policy can affect bank profits and their incentives to participate in the credit market. For instance, if $m_t^u > m_t^b$, which occurs when $\lambda^u = \lambda^b$ and $s > 0$, then an increase in s raises banks' profits.

Since banks' only interest-earning assets are the loans provided to entrepreneurs, and their liabilities do not bear interest, we identify the average net interest margin on a lending relationship as

$$NIM_t = \frac{\phi_t}{\lambda^b \left[\nu\left(k_t^b - m_t^b\right) + (1-\nu)\hat{k}\right]}, \tag{24}$$

where $\hat{k}$ is the solution to (18). The numerator are the fees paid by banked entrepreneurs while the denominator is the sum of all loans.

4.4 Creation of lending relationships

Free entry of banks in the market for relationship lending means $V_{t+1}^b \leq 0$, with equality if there is entry.

Lemma 1 *In any equilibrium where the market for relationship lending is active in all periods,* $\{\theta_t\}_{t=0}^{\infty}$*, solves*

$$\frac{\theta_t}{\alpha(\theta_t)} = \frac{\beta\eta\Delta\pi(s_{t+1})}{\zeta} - \beta(1-\eta)\theta_{t+1} + \beta(1-\delta)\frac{\theta_{t+1}}{\alpha(\theta_{t+1})}. \tag{25}$$

According to (25), monetary policy affects the creation of lending relationships through the term $\Delta\pi(s_{t+1})$. If $m^u_{t+1} > m^b_{t+1}$, e.g., if $\lambda^u = \lambda^b$, then an increase in s_{t+1} raises $\Delta\pi(s_{t+1})$ by reducing the net profits of unbanked entrepreneurs by more than the profits of banked entrepreneurs. This effect worsens the entrepreneur's status quo in the negotiation and raises ϕ_t. As a result, bank profits increase with s_{t+1}.

The measure of lending relationships at the start of a period evolves according to

$$\ell_{t+1} = (1-\delta)\ell_t + \alpha_t(1-\ell_t). \tag{26}$$

The number of lending relationships at the beginning of $t+1$ equals the measure of lending relationships at the beginning of t that have not been severed, $(1-\delta)\ell_t$, plus newly created relationships, $\alpha_t(1-\ell_t)$.

4.5 Equilibrium

An equilibrium with internal and external finance is a bounded sequence, $\{\theta_t, \ell_t, m^u_t, m^b_t, k^b_t, \hat{k}^b_t, \phi_t\}_{t=0}^{\infty}$, that solves (26), (15), (18), (19), (20), and (25) for a given $\ell_0 > 0$. In the following proposition we characterize equilibria when banked and unbanked entrepreneurs receive investment opportunities at the same frequency.

Proposition 2 ***(Equilibria with internal and external finance.)*** *Suppose* $\lambda^u = \lambda^b = \lambda$*. A unique steady-state monetary equilibrium exists and features an active credit market if and only if*

$$(\rho+\delta)\,\zeta < \eta\Delta\pi(s). \tag{27}$$

There are two regimes:

1. *Low spread regime:* $s_t \leq \hat{s} \equiv \lambda\psi'\left(\hat{k}\right)$*. All entrepreneurs who are not subject to theft invest* k_t *that solves (15). The difference in their asset holdings according to their banking status is*

$$m^u_t - m^b_t = \psi'^{-1}\left(\frac{s_t}{\lambda}\right). \tag{28}$$

In the neighborhood of $s = 0$*,* $\partial m^b/\partial s < \partial m^u/\partial s < 0$ *and* $\partial\theta/\partial s = 0$*.*

2. <u>High spread regime:</u> $s_t > \hat{s} \equiv \lambda\psi'\left(\hat{k}\right)$. Banked entrepreneurs hold no assets, $m_t^b = 0$, and invest $k_t^b = \hat{k}$.

<u>Transmission of monetary policy</u>. For all $s > 0$,

$$\frac{\partial\theta}{\partial s} = \eta\frac{\nu\left(m^u - m^b\right)}{\zeta}\left\{\frac{(\rho+\delta)\left[1-\epsilon(\theta)\right]}{\alpha(\theta)} + 1 - \eta\right\}^{-1} > 0 \ .$$

Proposition 2 distinguishes two regimes. If the cost of holding assets is low, between 0 and $\hat{s}$, both banked and unbanked entrepreneurs invest the same amount when they are not subject to theft. At one limit, when $s = 0$, internal finance is costless and, unless entrepreneurs are subject to theft, all investment opportunities are financed internally, $m_t^u = m_t^b = k_t^b = k^*$. At the other limit, when $s = \hat{s}$, entrepreneurs invest $\hat{k}$ which is financed internally by unbanked entrepreneurs and externally by banked entrepreneurs. In between 0 and $\hat{s}$, banked entrepreneurs make a down payment, $m_t^b < k_t$, and take a bank loan to cover the rest of their financing needs while unbanked entrepreneurs cover all investment expenditures with their liquid assets, $m_t^u = k_t$. In the special case when liquid assets are immune against theft, $\nu = 1$, (25) can be reexpressed as:

$$\frac{(\rho+\delta)\theta\zeta}{\alpha(\theta)} = \lambda\text{L}\left[NIM - \frac{\psi(\text{L})}{\text{L}}\right] \text{ with } NIM_t = \frac{\eta}{\lambda}s_t + \frac{(1-\eta)[\psi(\text{L}_t) - \zeta\theta_t/\lambda]}{\text{L}_t}.$$

The optimal creation of relationships by banks depends on the volume of loans, L, the NIM, and the unit cost of external finance, $\psi(\text{L})/\text{L}$. The NIM is composed of two terms. The first term is proportional to the interest rate spread. The second term is a function of monitoring and entry costs. Market tightness is driven to zero when s is sufficiently close to its lower bound.

In the second regime, when the cost of holding assets is larger than $\hat{s}$, banked entrepreneurs do not hold liquid wealth and resort to external finance only. If external finance is costless, $\psi \equiv 0$, then $\hat{s} = 0$ and only the second regime prevails. In that case, the equilibrium features $m_t^b = 0$ for all $s_t > 0$ and $k_t^b = \hat{k} = k^*$, i.e., investment levels are socially efficient. If external finance is costly, then $k^b > k^u$. We represent these two regimes in the left panel of Figure 4.

We now turn to the case where $\lambda^b > \lambda^u$. There are still two regimes depending on whether s is smaller or larger than $\hat{s} = \lambda^b\psi'\left(\hat{k}\right)$. If $s < \hat{s}$, all entrepreneurs hold some liquidity but the relationship between $m^u - m^b$ and s can be nonmonotone.

Proposition 3 ***(Money demands and banking status.)*** *Suppose the cost of external finance is $\psi(\text{L}) = \psi_0\text{L}^{1+\xi}/(1+\xi)$ and $\lambda^b > \lambda^u$. If $\xi < 1$ or $\xi = 1$ and $\left(\lambda^b - \lambda^u\right)/\lambda^u > -y''(k^*)/\psi_0$, then*

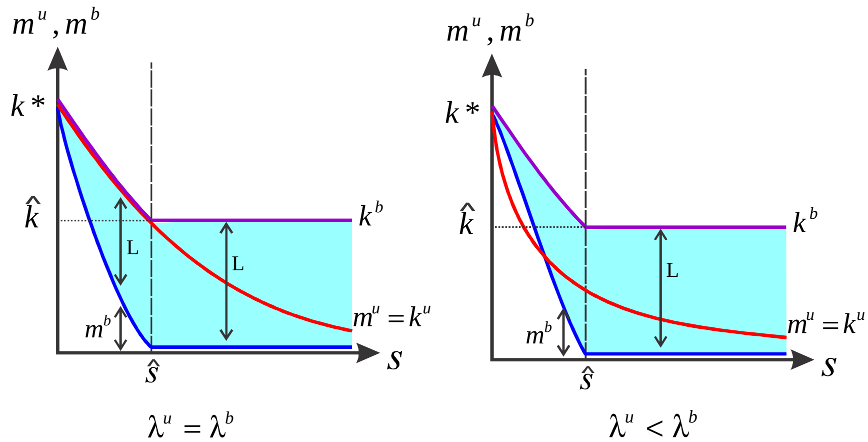

Figure 4: Holdings of liquid assets, loan sizes, and investment

there exists $0 < s_0 \leq s_1 < +\infty$ *such that: for all* $s < s_0$, $m^b > m^u$ *and* $\partial\theta/\partial s < 0$*; for all* $s > s_1$, $m^u > m^b$ *and* $\partial\theta/\partial s > 0$.

For sake of illustration, suppose the cost of external finance is quadratic, $\xi = 1$, and ψ_0 is sufficiently large. When s is low ($s < s_0$), it is optimal for banked entrepreneurs who receive more frequent investment opportunities to accumulate more liquid assets than unbanked entrepreneurs. In contrast, if s is large ($s > s_1$), banked entrepreneurs can rely on bank credit to finance investment, and hence they hold less liquid assets than unbanked entrepreneurs. The crossing of the money demands, m^u (red curve) and m^b (blue curve), is illustrated in the right panel of Figure 4. Using that $\partial\theta/\partial s \sim m^u - m^b$, the nonmonotonicity of $m^u - m^b$ with respect to s creates a nonmonotone relationship between bank entry and spreads. For low spreads, an increase in s reduces bank entry whereas for high spreads it raises bank entry.

In summary, our model delivers a transmission mechanism from the policy rate to investment through two channels. There is an internal finance channel whereby an increase in s_t reduces entrepreneurs' holdings of liquid assets, which in turn reduces the share of total investment financed internally. This effect is asymmetric for banked and unbanked entrepreneurs and depends on the elasticity of $\psi(\text{L})$ and the difference between λ^b and λ^u. There is also an external finance channel according to which an increase in s makes lending relationships more valuable when $m^u > m^b$, which raises bank profits when banks have market power and promotes loan creations.

5 Aftermath of a credit shock

We now study the dynamics of the economy following a credit supply shock under alternative monetary policies. In order to simplify the presentation, we set $\nu = 1$ for now and we reintroduce $\nu < 1$ when we calibrate the model and turn to the optimal policy analysis. The economy starts at a steady state with $m_0^u > m_0^b$ (which holds if ψ is sufficiently convex or s is above a threshold) and $\ell_0 = \ell^s$. A banking crisis destroys a fraction of the lending relationships, $\ell_0^+ < \ell^s$. We illustrate the dynamics in a phase diagram for the continuous-time limit of our model (see the Appendix for the derivations).[17]

5.1 Interest or liquidity targeting vs. forward guidance

We consider simple policies that illustrate the central trade-off of the policymaker between providing liquidity to unbanked entrepreneurs and promoting the creation of lending relationships.

Interest rate targeting. The first policy consists in keeping the spread, s_t, constant over time so as to maintain an elastic supply of liquidity. The steady state is a saddle point with a unique saddle path, $\theta_t = \theta^s$ for all t, leading to it, as illustrated by the left panel of Figure 5. For any ℓ_0, the measure of relationships is given by

$$\ell_t = \ell^s + (\ell_0 - \ell^s)\, e^{-[\delta+\alpha(\theta^s)]t},$$

where $\ell^s = \alpha(\theta^s)/\left[\delta + \alpha(\theta^s)\right]$. The speed of recovery, $\delta + \alpha(\theta^s)$, increases with the interest rate spread, s. Aggregate liquidity is given by $\mathrm{M}_t = \ell_t m^b + (1-\ell_t)m^u$, i.e.,

$$\mathrm{M}_t = \mathrm{M}^s + (\ell^s - \ell_0)\, e^{-[\delta+\alpha(\theta^s)]t}(m^u - m^b),$$

where $\mathrm{M}^s = \ell^s m^b + (1-\ell^s)m^u$. Hence, aggregate liquidity jumps upward as the credit shock occurs, and returns gradually to its steady-state value over time. Investment levels by banked and unbanked firms are unaffected by the shock, $k_t^u = k_0^u < k_t^b = k_0^b$, as illustrated in the bottom right panel of Figure 5. Aggregate investment falls since the measure of unbanked firms is higher relative to the steady state and those firms can only finance a fraction of the investment opportunities of banked firms ($\lambda^u < \lambda^b$).

[17] For a detailed description of New Monetarist models in continuous time, with methods and applications, see Choi and Rocheteau (2019).

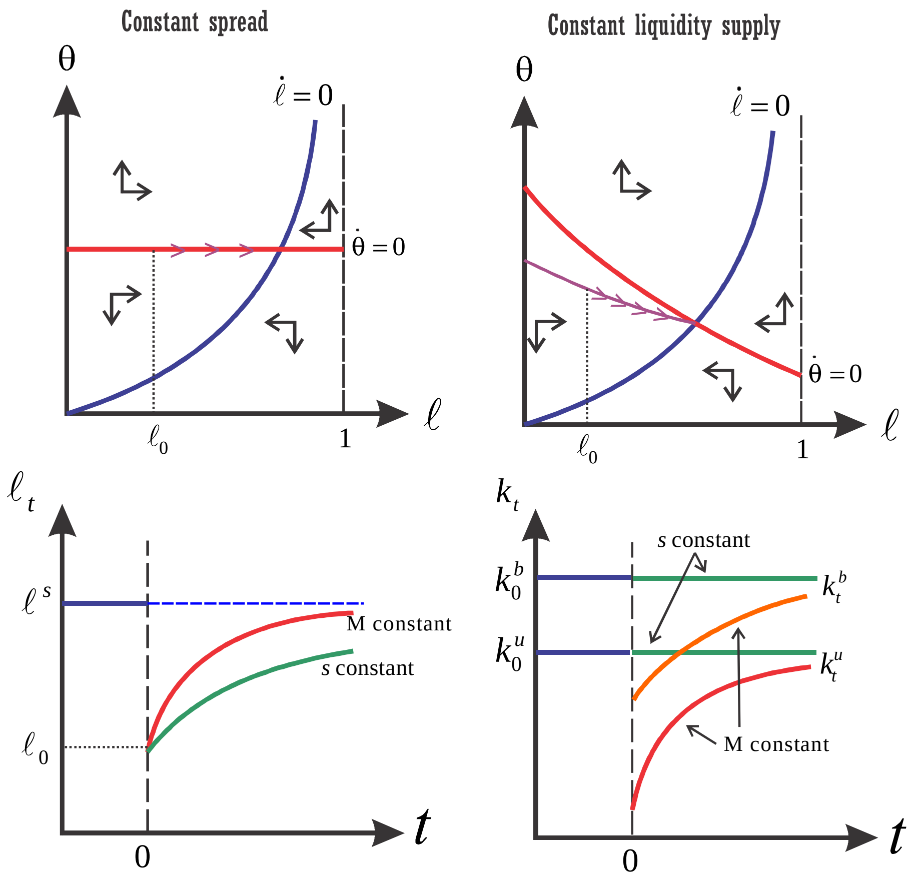

Figure 5: Transitional dynamics: phase diagram with constant s (top left) and constant M (top right); dynamics of lending relationships (bottom left) and investment (bottom right)

Aggregate liquidity targeting. Suppose next that aggregate liquidity, M_t, is held constant. The market-clearing spread, $s(\ell, \mathrm{M})$, is decreasing in both ℓ and M for all ℓ and $\mathrm{M} < k^*$ and $s(\ell,\mathrm{M}) = 0$ for all $\mathrm{M} > k^*$. The θ-isocline is now downward sloping in (ℓ, θ)-space when $\mathrm{M} < k^*$ and is horizontal if $\mathrm{M} > k^*$. Intuitively, as ℓ increases, the aggregate demand for liquid assets decreases, and hence s decreases, which reduces the profitability of banks and bank entry. The steady state is a saddle point, and the saddle path is downward sloping if $\mathrm{M} < k^*$, as illustrated by the top right panel of Figure 5.

If $\mathrm{M} < k^*$, the interest rate spread and credit market tightness increase at the time of the credit crunch. As the economy recovers, both θ and s decrease and gradually return to their steady-state values. Investment by banked and unbanked firms drops initially, since s is higher, but recovers afterwards. So, keeping M constant speeds up the formation of lending relationships (see bottom left panel of Figure 5), but does not accommodate the higher demand for liquidity created by the larger fraction of unbanked entrepreneurs, thereby reducing individual investment.

Forward guidance. The policymaker faces a trade off between providing liquidity to unbanked entrepreneurs by keeping spreads low and giving incentives to banks to re-enter and rebuild relationships by raising spreads. In order to address both objectives, the policymaker can take advantage of banks' dynamic incentives by setting a low interest spread initially, $s(t) = s_L$ for all $t < T$, to allow unbanked entrepreneurs to self insure at low cost, and by committing to raise the interest spread at some future date T, i.e., $s(t) = s_H > s_L$ for all $t > T$. We illustrate this policy in the top right panel of Figure 5. The low-spread regime corresponds to a low θ-isocline ($\theta = \theta_L$) and the high-spread regime corresponds to a high θ-isocline ($\theta = \theta_H$). The arrows of motion characterize the dynamic system when $t < T$. The equilibrium path can be obtained by moving backward in time.[18] For all $t > T$, the economy is on the horizontal saddle path corresponding to $s = s_H$, i.e., $\theta_t = \theta_H$. The path for the economy is continuous at $t = T$, i.e., $\theta_{T^-} = \theta_H$, and it reaches the θ_H-saddle path by below. Finally, the trajectory of the economy starts at ℓ_0 with $\theta_0 > \theta_L$. Over the time interval $(0, T)$, market tightness rises until it reaches θ_H at time T. Aggregate liquidity increases initially due to high demand and low spreads, decreases gradually as ℓ_t increases, and jumps downward when the spread is raised.

5.2 Calibration

We calibrate the model to match moments on U.S. small businesses and their banking relationships. We use data from the 2003 National Survey of Small Business Finances (SSBF) and Compustat when the SSBF is insufficient. We take the period length as a month and set $\rho = (1.04)^{\frac{1}{12}} - 1 = 0.0033$. We adopt the following functional forms: $\alpha(\theta) = \overline{\alpha}\theta/(1+\theta)$, where $\overline{\alpha} \in [0,1]$, $y(k) = k^a/a$, and $\psi(\mathrm{L}) = B\mathrm{L}^{1+\xi}/(1+\xi)$, where $\xi > 1$. The parameters to calibrate are then $(\bar{\alpha}, \delta, \xi, B, a, \lambda^u, \lambda^b, s, \nu, \eta, \zeta)$.

[18] Choi and Rocheteau (2019) describe the methodology to solve for equilibria of a continuous-time New Monetarist model with policy announcements.

Parameter	Value	Moment	Data	Model
Parameters Set Directly				
Discount rate (annual %), ρ	4.00			
Destruction rate, δ	0.01	Avg. length of credit rel. (years)	5.61	5.61
Probability of theft, $1-\nu$	0.0016	Rate of embezzlement and forgery	1.9	1.9
Parameters Set Jointly		**Moments used in optimization**		
Matching efficiency, $\bar{\alpha}$	0.093	Share of banked firms	0.68	0.68
Production curvature, a	0.555	Elasticity of cts^u to s	-0.23	-0.23
Productivity shock, unbanked, λ^u	0.126	Average cash-to-sales, unbanked	0.36	0.36
Productivity shock, banked, λ^b	0.144	Effect of relationship on innovation	1.14	1.14
External finance - curvature, ξ	8.604	Elasticity of $(cts^u - cts^b)$ to s	0.07	0.07
External finance level, B	6.187	$(cts^u - cts^b)/cts^u$	0.58	0.58
Bargaining power, η	0.522	Average NIM (%)	6.00	6.00
Bank entry cost, ζ	0.025	Optimal spread (% annual)	2.00	2.00

We define a credit relationship as an open line of credit or an active business credit card with a bank. In the 2003 SSBF, 68% of small businesses report being in a credit relationship with a bank, with an average duration of 8.25 years. We set $\bar{\alpha} = 0.09$ and $\delta = 1/99$ to generate $\ell = 0.68$ and an average length of relationship of 99 months.[19]

We interpret s as the user cost of holding an index of money-like assets, MSI-ALL, which includes currency, deposit accounts, and institutional and retail money market funds.[20] As our baseline, we target the optimal long-run spread under commitment to equal the average real user cost of MSI-ALL from 2002–2004 of 2%. This pins down the fixed cost of bank entry and gives $\zeta = 0.025$.

We set (λ^u, a) to target the level and elasticity of cash-to-sales by the unbanked with respect to the user cost, illustrated in Figure 1. In the model, annual cash-to-sales is given by $cts^u = m^u/[12 \cdot \lambda^u \nu y(m^u)]$. In the data, we rely on evidence from unbanked firms in Compustat since firms in the SSBF do not consistently report sales for much of the sample.[21] This procedure yields $\lambda^u = 0.126$ and $a = 0.56$. We then set λ^b based on the evidence in Cosci et al. (2016) who find that being in a lending relationship increases the probability of innovation or R&D activity by a factor of 1.14. Hence, $\lambda^b = 1.14 \times \lambda^u = 0.144$.

[19] Similarly, Sufi (2009) finds 74.5% of U.S. public, non-financial firms in Compustat have access to lines of credit provided by banks.

[20] The user cost is the spread between the rate of return of the MSI-ALL index and a benchmark rate equal to the maximum rate across short-term money market assets plus a liquidity premium of 100 basis points. See Anderson and Jones (2010) for details on constructing the MSI-ALL series and the rate of return and FRED series OCALLP.

[21] Additionally, Compustat delivers a longer time series from 1996 to 2003 versus the shorter variation in the SSBF from 2003 to 2005 which is important for generating reliable money demand estimates.

We set (B, ξ) to target the level and elasticity of the difference in cash-to-sales between unbanked and banked firms, $(cts^u - cts^b)$, with respect to s. As discussed in Section 2, unbanked small businesses hold 58% more cash relative to sales. We set B to target $(cts^u - cts^b)/cts^u = 0.58$ and ξ to target the elasticity of $(cts^u - cts^b)$ with respect to s of 0.072. This implies $(B, \xi) = (6.19, 8.60)$.

We set ν to match evidence on the incidence of corporate embezzlement and forgery from the FBI's Uniform Crime Reporting data. During 2017, 114,110 incidents of embezzlement and forgery were reported to the FBI.[22] The U.S. Census Statistics of U.S. Businesses reports a total of 5,996,900 employer firms, which implies an annual incidence of 1.9% or a monthly incidence of 0.16%. Hence we set $\nu = 0.9984$.

Finally, we set banks' bargaining power, η, to target the average annual NIM on small business loans of 6%, as illustrated in right panel of Figure 2.[23] This procedure gives $\eta = 0.52$.

Response to destruction of relationships. We consider different magnitudes for the size of the shock: a contraction in ℓ of 10%, 35%, and 60%. In our calibrated economy, ℓ falls from a steady-state of 0.68 to 0.59, 0.43 and 0.26, respectively. These shocks correspond to different interpretations for the contraction in small business lending in the U.S. during the 2008 banking crisis and recession.[24]

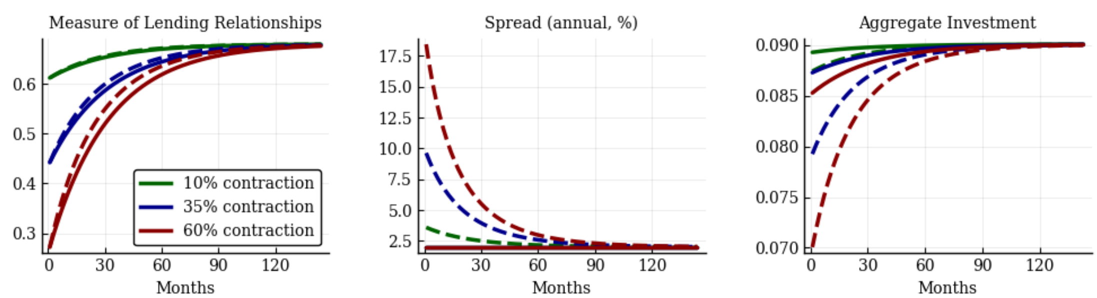

Figure 6: Dynamic response to destruction of lending relationships: constant spread (solid) vs. constant liquidity (dashed)

[22] See https://ucr.fbi.gov/nibrs/2017/tables/data-tables.

[23] Our measure of NIM from (24) is interpreted as the discounted sum of net interest margins over the duration of the loan. By setting NIM = 6% to its value at an annual frequency, we implicitly interpret the maturity of a loan in the model as a year, thereby aligning it to the frequency of investment opportunities. If the average maturity is longer than a year, the per loan NIM will be higher, and vice versa. To check the sensitivity of our results to this target we consider cases in which we either half or double the baseline value of banks' bargaining power, η.

[24] The contraction in lending relationships of 10% is in line with evidence from McCord and Prescott (2014) of a 14% decline in the number of commercial banks from 2007 to 2013. The larger contractions of 35% and 60% correspond roughly to the fall in the measure of small business loan originations of 40% reported in Chen, Hanson, and Stein (2017) and the fall in the total number of U.S. corporate loans of 60% as reported in Ivashina and Scharfstein (2010).

Figure 6 shows the dynamic response of the calibrated model under two fixed monetary policy rules: the spread s remains constant over time (solid lines) and the aggregate supply of liquidity M remains constant (dashed lines). Those dynamics are qualitatively similar to those in Figure 5. Quantitatively, the policy that consists in keeping M constant generates a large decline in aggregate investment (by about 20% under the large shock) which is more than twice as large as the one obtained under a constant spread. The recovery in terms of lending relationships is faster, i.e., the half time is reached about 7 months earlier under a constant M than under a constant s.

6 Optimal monetary policy

We now analyze the optimal monetary policy following a credit crisis that destroys a fraction of lending relationships. In the Appendix, we show any constrained-efficient allocation has $k_t^u = k_t^b = k^*$, $\mathrm{L}_t = 0$, $k_t^b = \hat{\mathrm{L}}_t = \hat{k}$. It coincides with a decentralized equilibrium if the policymaker implements the Friedman rule, $s_t = 0$, to achieve optimal investment levels, and the Hosios condition holds, $\epsilon(\theta_t) = \eta$, to guarantee an efficient creation of lending relationships.[25] In the Appendix, we also establish sufficient conditions under which the Friedman rule, $s_t = 0$ for all t, is suboptimal. If the difference between $\epsilon(\theta)$ and η is sufficiently large relative to $1/\xi$, increasing s_{t+1} above 0 raises the creation of relationships which is inefficiently low in the decentralized equilibrium, and the associated welfare gain outweighs the monitoring cost of bank loans.[26]

6.1 The Ramsey problem

Suppose the policymaker chooses an infinite sequence of interest spreads to implement a decentralized equilibrium that maximizes social welfare. The policy path, $\{s_t\}_{t=1}^{\infty}$, is announced before the market for relationships opens in stage 2, and the policymaker commits to it.[27] We write the Ramsey problem recursively by treating credit market tightness in every period $t \geq 1$, θ_t, as a state variable. The initial tightness, θ_0, is chosen to place the economy on the optimal path. Market tightness, θ_t, is interpreted as a promise to banks that determines their future profits. It must be honored in period $t+1$ by choosing s_{t+1} and θ_{t+1} consistent with the free entry condition, (25).

[25] See the appendix for a formal proof. For a related result, see Berentsen, Rocheteau, and Shi (2007).

[26] We do not allow the policymaker to make direct transfers to banks that participate in the credit market in order to correct for inefficiently low entry. Such transfers may not be feasible if the policymaker cannot distinguish between active and inactive banks in the credit market (i.e., one could create a bank but not search actively in the credit market).

[27] We also consider commitment under a timeless approach based on Woodford (1999, 2003) in the Appendix.

The recursive planner's problem is

$$\begin{aligned}\widetilde{\mathbb{W}}(\ell_t,\theta_t) \quad = \quad & \max_{\theta_{t+1}\in\Gamma(\theta_t)} \left\{-\zeta\theta_t(1-\ell_t)+\beta(1-\ell_{t+1})\lambda^u\nu\left[y(m^u_{t+1})-m^u_{t+1}\right]\right. \\ & +\beta\ell_{t+1}\lambda^b\left\{\nu\left[y\left(k^b_{t+1}\right)-k^b_{t+1}-\psi(\mathrm{L}_{t+1})\right]+(1-\nu)\left[y\left(\hat{k}\right)-\hat{k}-\psi(\hat{k})\right]\right\} \\ & \left.+\,\beta\widetilde{\mathbb{W}}(\ell_{t+1},\theta_{t+1})\right\}, \end{aligned} \tag{29}$$

where $\Gamma(\theta_t)$ is the set of values for θ_{t+1} consistent with (25) for some $s_{t+1}\in[0,+\infty)$. Given s_{t+1}, the quantities m^u_{t+1}, k^b_{t+1}, and L_{t+1} are obtained from (15) and (19).[28] The policymaker's problem at the beginning of time is

$$\mathbb{W}(\ell_0)=\max_{\theta_0\in\Omega=\left[\underline{\theta},\bar{\theta}\right]}\widetilde{\mathbb{W}}(\ell_0,\theta_0), \tag{30}$$

where $\underline{\theta}$ (respectively, $\bar{\theta}$) is the steady-state value of θ if $s=0$ (respectively, $s=\infty$).

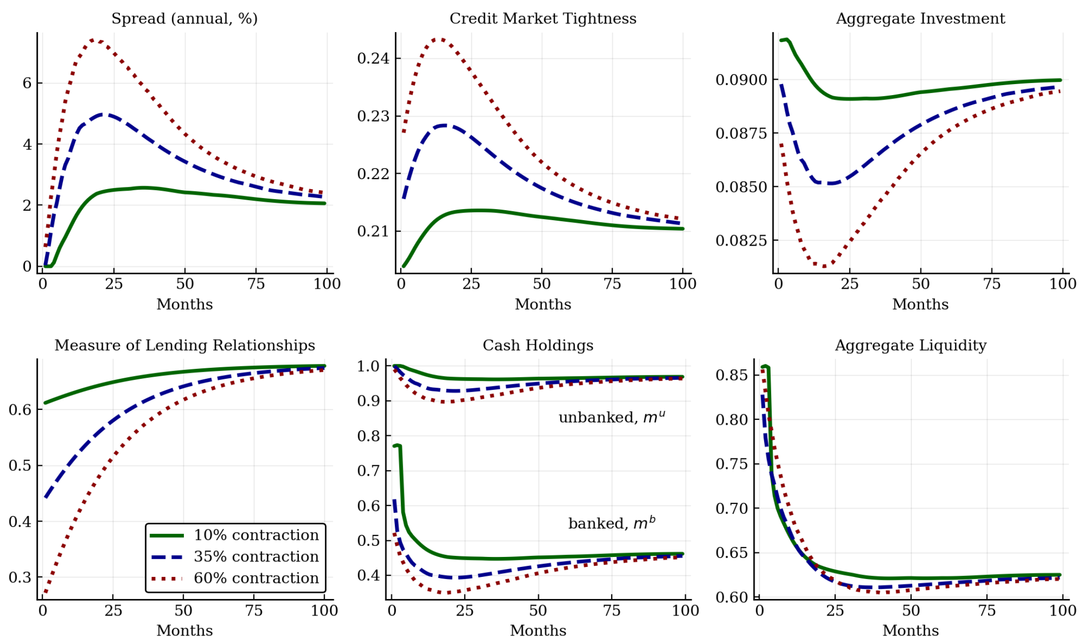

Figure 7: Optimal policy response with commitment to a destruction of lending relationships

Figure 7 illustrates the optimal policy response following a banking shock that destroys lending relationships in the calibrated economy. The Ramsey solution lowers s_t close to the lower bound at the onset of the crisis and then raises it above its long-run value (forward guidance) before gradually decreasing it to the stationary level. The hump-shaped path of the optimal spread (top left panel of Figure 7) causes a similar hump-shaped response in credit market tightness (top middle panel). While low spreads benefit unbanked firms by making internal finance less costly, they dampen bank

[28] The existence and uniqueness of $\widetilde{\mathbb{W}}(\ell,\theta)$ is established in the Appendix.

profits and reduce the incentives to create new relationships. The use of forward guidance mitigates the effect of the current spread on relationship creation because banks' lifetime profits depend on the whole path of future spreads. For smaller shocks, initial spreads are lower for longer and only gradually increase over time. However, for large shocks, spreads increase more rapidly and feature a more pronounced hump-shaped response.

Low spreads induce all firms to raise their holdings of liquid assets closer to the full insurance level $k^* = 1$ (bottom middle). Over time, the policymaker gradually unwinds the initial expansion of M, leading to a sharp decline in banked firms' cash holdings relative to unbanked firms. Initially, the increase in liquidity goes beyond what is necessary to keep aggregate investment constant (top right). Afterwards, aggregate investment falls and reaches its lowest value after about 16 months. It recovers gradually as it increases towards its long-run value.

6.2 Optimal policy without commitment

We now relax the assumption of commitment altogether and assume the policymaker sets s_{t+1} in period t but cannot commit to $\{s_{t'}\}_{t'>t+1}$.[29] As in Klein, Krusell, and Rios-Rull (2008), the policymaker moves first by choosing s_{t+1}, and the private sector moves next by choosing θ_{t+1}, m^u_{t+1}, and m^b_{t+1}.

We restrict our attention to Markov perfect equilibria. The policymaker's strategy consists of a spread s_{t+1} at the beginning of stage 2 of period t as a function of the economy's state, ℓ_t. From (15), $m^u_t = y'^{-1}\left[1 + s_t/\lambda^u\right]$, so the strategy of the policymaker can be represented by $m^u_{t+1} = \mathcal{K}(\ell_t)$. The strategy of banks to enter is expressed as $\theta_t = \Theta(\ell_t, m^u_{t+1})$, where Θ is implicitly defined by (25), i.e.,

$$\frac{\Theta(\ell_t, m^u_{t+1})}{\alpha\left[\Theta(\ell_t, m^u_{t+1})\right]} = \frac{\beta\eta\Delta\pi(m^u_{t+1})}{\zeta} - \beta(1-\eta)\Theta(\ell_{t+1}, m^u_{t+2}) + \beta(1-\delta)\frac{\Theta(\ell_{t+1}, m^u_{t+2})}{\alpha\left[\Theta(\ell_{t+1}, m^u_{t+2})\right]}, \tag{31}$$

where $\ell_{t+1} = (1-\delta)\ell_t + \alpha\left[\Theta(\ell_t, m^u_{t+1})\right](1-\ell_t)$ and $m^u_{t+2} = \mathcal{K}(\ell_{t+1})$. When forming expectations about θ_{t+1}, banks anticipate the policymaker in period $t+1$ will adhere to his policy rule, $m^u_{t+2} = \mathcal{K}(\ell_{t+1})$, and hence $\theta_{t+1} = \Theta\left[\ell_{t+1}, \mathcal{K}(\ell_{t+1})\right]$. In equilibrium, $\theta_t = \Theta(\ell_t, m^u_{t+1})$ and $m^u_{t+1} = \mathcal{K}(\ell_t)$ are best responses to each other.

[29] As an alternative to relaxing the commitment assumption altogether, we consider in the Appendix the timeless approach to the Ramsey problem as in Woodford (2003).

Given Θ, we determine $\mathcal{K}(\ell_t)$ recursively from:

$$\mathcal{W}(\ell_t) = \max_{m^u_{t+1}\in[0,k^*]} \left\{-\zeta(1-\ell_t)\Theta\left(\ell_t, m^u_{t+1}\right) + \beta(1-\ell_{t+1})\lambda^u\nu\left[y(m^u_{t+1}) - m^u_{t+1}\right]\right. \tag{32}$$

$$+\beta\ell_{t+1}\lambda^b\left\{\nu\left[y\left(k^b_{t+1}\right) - k^b_{t+1} - \psi(\mathrm{L}_{t+1})\right] + (1-\nu)\left[y\left(\hat{k}\right) - \hat{k} - \psi(\hat{k})\right]\right\}$$

$$\left. +\beta\mathcal{W}(\ell_{t+1})\right\}, \tag{33}$$

where k^b_{t+1} and L_{t+1} can be expressed as functions of m^u_{t+1}. The entire transitional dynamics are computed numerically by devising a two-dimensional iteration described in the Appendix. Figure 8 illustrates the optimal policy response under the benchmark calibration.

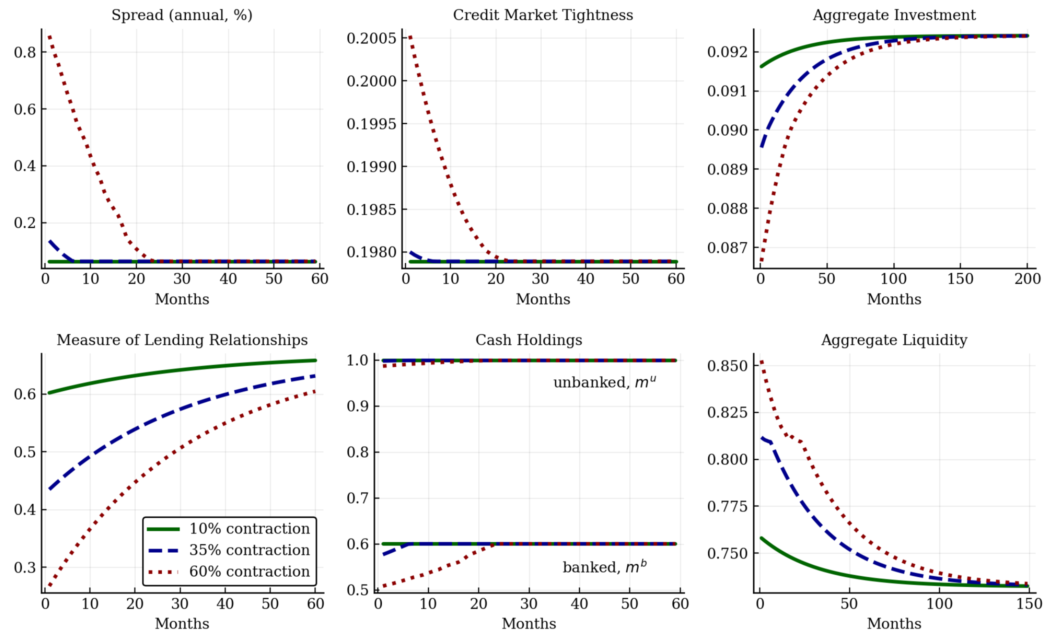

Figure 8: Optimal policy response without commitment to a destruction of lending relationships

The policymaker can no longer commit to raise future s_t to counteract the effects of current s_t on banks' lifetime expected profits, as in the Ramsey problem. In the long run, it is optimal to set spreads at their lower bound, zero, which illustrates the bias of the policymaker toward low interest rates. When the shock to banking relationships is small, the optimal policy is to keep spreads unchanged at their optimal long-run level (top left panel of Figure 8). However, for large enough shocks, the policymaker raises spreads at the outset of the credit crunch when ℓ_t is low in order to rebuild relationships more quickly. As lending relationships recover, s_t is reduced gradually over time. Quantitatively, interest spreads early on are larger than the ones from the Ramsey solution when the shock is large enough, but future spreads are lower.

There are two effects on aggregate liquidity at the onset of the crisis. The increase in spreads lowers holdings of liquid assets for all firms, thereby decreasing M_t. However the fall in ℓ from the credit shock increases M_t since $m^u > m^b$. Quantitatively, the second effect tends to dominate (bottom right). In contrast to the Ramsey solution, aggregate investment falls at the time of the shock and recovers gradually (top right).

In our calibrated example, the half life of the recovery following a 60% contraction is 20 months under the Ramsey solution but 27 months under the no commitment policy. So, the inability to commit slows down the recovery by about 7 months. The welfare loss from the lack of commitment ranges from 0.90% to 0.98% of foregone output, depending on the size of the shock.[30] These estimates are consistent with those in Klein, Krussell, and Rios-Rull (2008) who study time consistent capital income taxation and find that without commitment, steady-state consumption is about 0.5% lower relative to the commitment case.

6.3 Optimal Policy following a temporary lockdown

We conclude this section on optimal monetary policy by analyzing the response to an unanticipated, but temporary, business lock-down. The shock aims to capture the COVID-19 pandemic that has forced a large fraction of small businesses to shut down operations. Our model is able to speak to an aspect of the crisis where large disruptions in firms' production reduces the incentive for banks to create new lending relationships while existing relationships are destroyed at a constant rate, with potential consequences for the recovery. To capture the temporary lock-down, we assume the rates of production opportunities, λ^u and λ^b, unexpectedly fall by 40%, based on a recent report (U.S. Chamber of Commerce 2020) that 25% to 55% of small businesses either closed or would likely close as a result of COVID-19.[31] We consider three scenarios where the lock-down lasts three, six, or twelve months, then returns to its normal, pre-pandemic levels.

Figure 9 plots the optimal response when the policy maker has commitment. The top left panel illustrates the disruption in production opportunities faced by the unbanked (the disruption faced by the banked looks similar). If the disruption is anticipated to last three months (green lines), the response consists of initially lowering s_t close to the lower bound (top middle panel) and keeping

[30] We measure the welfare cost of not implementing the Ramsey solution as $1 - \mathcal{W}^p/\mathcal{W}^R$, where $\mathcal{W}^R$ is lifetime discounted output under the Ramsey policy and $\mathcal{W}^p$ is lifetime discounted output under the optimal policy with no commitment. In the Appendix, we report robustness checks on various parameters (e.g., ν) and summarize how this welfare cost is affected.

[31] Similarly, Bartik et al. (2020) surveyed small businesses and found 43% of them were temporarily closed.

the spread low past when the shock is over, only gradually returning to normal after around 20 months. Despite low spreads, θ_t falls initially but begins to rise before the shock is over. Banks anticipate that production activity will return to normal and therefore have incentives to create long lasting relationships before the recovery materializes. Aggregate investment (bottom right) overshoots its long-run level upon recovery.

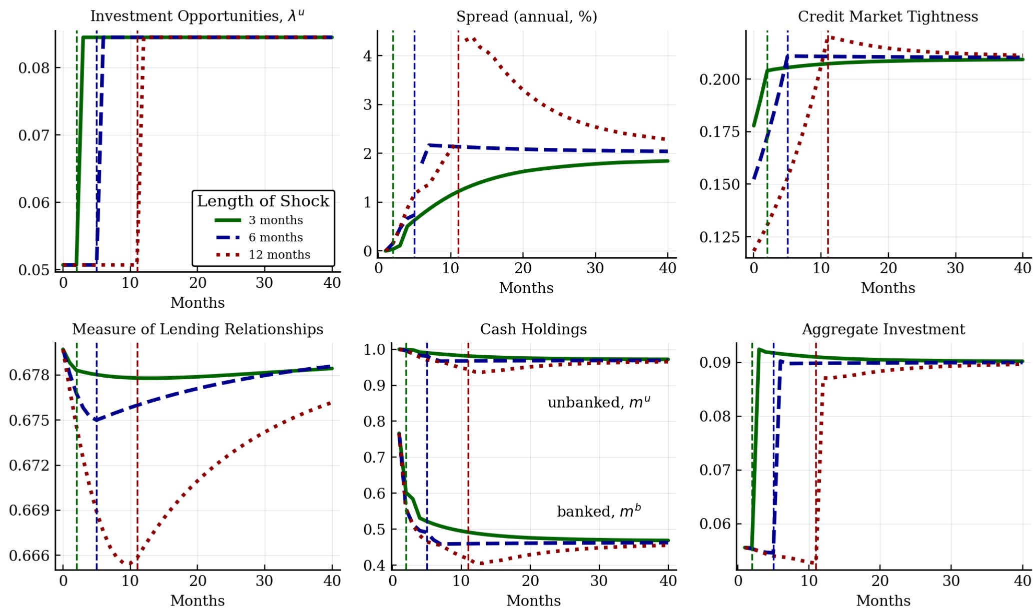

Figure 9: Optimal policy response to a temporary reduction in λ_u and λ_b by 40%

When the length of the disruption becomes longer at six or twelve months (dashed blue and dotted red lines in Figure 9) optimal policy still features low initial spreads, but the policymaker increases spreads gradually before the lock-down is over and increases them discontinuously at the time of the recovery, above their long-run value. High future spreads prevent tightness and the measure of lending relationships from falling too low. As a result, aggregate investment (bottom left) returns to normal levels as soon as the lock-down ends.

7 Conclusion

In this paper, we argue that the formation of lending relationships is critical for small businesses to finance their investment opportunities. As the formation of these relationships can be influenced by monetary policy, we developed a general equilibrium model of corporate finance that formalizes this transmission mechanism, building on recent theories of money demand under idiosyncratic risk and financial intermediation in over-the-counter markets. We use our model to study the optimal response of the monetary authority following a banking crisis described as an exogenous destruction of a fraction of the existing lending relationships. We consider different assumptions regarding the policymaker's power to commit to setting a time path of interest spreads.

If the policymaker can commit over an infinite time horizon, the optimal policy involves "forward guidance": the interest spread is set close to its lower bound at the outset of the crisis and increases over time as the economy recovers. It is this promise of high future spreads that provides banks incentives to keep creating lending relationships, even in a low spread environment. However, such promises are not time consistent. If the policymaker cannot commit more than one period ahead, then the interest rate spread is persistently low and the recession is more prolonged.

Our model of lending relationships and corporate finance can be extended in several ways. For instance, one could relax the assumption that banks can fully enforce repayment to study imperfect pledgability of firms' returns and its relation with monetary policy (e.g., as in Rocheteau, Wright, and Zhang 2018). In addition, while we assume full commitment by banks, one could consider banks' limited commitment and analyze the dynamic contracting problem in the credit market (e.g., as in Bethune, Hu, and Rocheteau 2017) or agency problems between firms and banks to capture additional benefits of lending relationships (e.g., Hachem 2011, Boualam 2017). It would be fruitful to develop a life-cycle version of our model to explain firms' cash accumulation patterns and their interaction with long term credit lines. Finally, our model of relationship lending could be applied in other institutional contexts, like the interbank market (Brauning and Fecht 2016).

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Appendix A1: Proofs of Propositions and Lemmas

Proof of Proposition 1. We first compute the surplus of being in a lending relationship for entrepreneurs and banks. Recall from (4) and (5) that the lifetime expected utilities of banked and unbanked entrepreneurs with wealth ω in the last stage of period t solve:

$$W_t^e(\omega) = \omega - \frac{m_{t+1}^u}{1+r_{t+1}} + \beta U_{t+1}^e(m_{t+1}^u), \tag{34}$$

$$X_t^e(\omega) = \omega - \frac{m_{t+1}^b}{1+r_{t+1}} + \beta Z_{t+1}^e(m_{t+1}^b). \tag{35}$$

The surplus of a banked entrepreneur is defined as $\mathcal{S}_t^e = \left[X_{t-1}^e(0) - W_{t-1}^e(0)\right]/\beta$. Substituting Z_t^e by its expression given by (11), i.e.,

$$Z_t^e(m_t^b) = \nu m_t^b - \phi_t + \lambda^b \nu \left[y(k_t^b) - k_t^b\right] + \lambda^b(1-\nu)\left[y(\hat{k}_t^b) - \hat{k}_t^b\right] + \delta W_t^e(0) + (1-\delta)X_t^e(0),$$

into (35) and subtracting $W_{t-1}^e(0)$, we obtain:

$$\begin{aligned}\beta\mathcal{S}_t^e &= \beta\left\{-\nu s_t m_t^b - \phi_t + \lambda^b\nu\left[y(k_t^b) - k_t^b\right] + \lambda^b(1-\nu)\left[y(\hat{k}_t^b) - \hat{k}_t^b\right]\right\} \\ &\quad +(1-\delta)\beta\left[X_t^e(0) - W_t^e(0)\right] + \beta W_t^e(0) - W_{t-1}^e(0).\end{aligned} \tag{36}$$

From (9), for all $m_t^u \le k^*$,

$$U_t^e(m_t^u) = \lambda^u\nu\left[y(m_t^u) - m_t^u\right] + \nu m_t^u + V_t^e(0),$$

which we substitute into (34) to express $W_{t-1}^e(0)$ as:

$$W_{t-1}^e(0) = \beta\{-\nu s_t m_t^u + \lambda^u\nu\left[y(m_t^u) - m_t^u\right]\} + \beta V_t^e(0).$$

Substituting $W_{t-1}^e(0)$ into (36) and dividing both sides by β we obtain:

$$\begin{aligned}\mathcal{S}_t^e &= -\phi_t - \nu s_t m_t^b + \lambda^b\nu\left[y(k_t^b) - k_t^b\right] + \lambda^b(1-\nu)\left[y(\hat{k}_t^b) - \hat{k}_t^b\right] \\ &\quad - \left[\pi_t^u + V_t^e(0) - W_t^e(0)\right] + (1-\delta)\beta\mathcal{S}_{t+1}^e.\end{aligned} \tag{37}$$

From (12), the surplus of the bank solves

$$\mathcal{S}_t^b = \phi_t - \lambda^b\nu\psi\left(k_t^b - m_t^b\right) - \lambda^b(1-\nu)\psi\left(\hat{k}_t^b\right) + \beta(1-\delta)S_{t+1}^b, \tag{38}$$

where m_t^b is the entrepreneur's down payment on the loan and hence $\mathrm{L}_t = k_t^b - m_t^b$ is the loan size. Summing (37) and (38), the total surplus of a lending relationship, $\mathcal{S}_t = \mathcal{S}_t^e + \mathcal{S}_t^b$, solves

$$\begin{aligned}\mathcal{S}_t &= -\nu s_t m_t^b + \lambda^b\nu\left[y(k_t^b) - k_t^b - \psi\left(k_t^b - m_t^b\right)\right] \\ &\quad +\lambda^b(1-\nu)\left[y(\hat{k}_t^b) - \hat{k}_t^b - \psi\left(\hat{k}_t^b\right)\right] \\ &\quad - \left[\pi_t^u + V_t^e(0) - W_t^e(0)\right] + (1-\delta)\beta\mathcal{S}_{t+1}.\end{aligned} \tag{39}$$

A lending relationship contract negotiated at time $t-1$ is

$$\{k_{t+\tau}^b, \hat{k}_{t+\tau}^b, m_{t+\tau}^b, \phi_{t+\tau}\}_{\tau=0}^{\infty} \in \arg\max[\mathcal{S}_t^b]^\eta[\mathcal{S}_t^e]^{1-\eta}. \tag{40}$$

Given the linearity of $\mathcal{S}_t^e$ and $\mathcal{S}_t^b$ in ϕ_t, a solution is such that $\{k_{t+\tau}^b, \hat{k}_{t+\tau}^b, m_{t+\tau}^b\}_{\tau=0}^{\infty} \in \arg\max \left\{\mathcal{S}_t^b + \mathcal{S}_t^e\right\}$ and $\phi_t \in \arg\max[\mathcal{S}_t^b]^{\eta}[\mathcal{S}_t^e]^{1-\eta}$. The maximization of the total surplus implies $\hat{k}_t^b = \hat{k}$ such that $y'(\hat{k}) = 1 + \psi'\left(\hat{k}\right)$. Maximizing $\mathcal{S}_t$ with respect to k_t^b and m_t^b gives:

$$\begin{aligned} y'(k_t^b) - 1 - \psi'\left(k_t^b - m_t^b\right) &= 0 \\ -\nu s_t + \lambda^b \nu \psi'\left(k_t^b - m_t^b\right) &\leq 0, \quad \text{“=” if } m_t^b > 0. \end{aligned}$$

The sequence of intermediation fees solves $\mathcal{S}_{t+\tau}^b = \eta \mathcal{S}_{t+\tau}$ for all τ, where $\mathcal{S}_t^b$ obeys (38). Using the definition of $\pi^b(s_t)$ in (21) we can reexpress $\mathcal{S}_t$ as:

$$\mathcal{S}_t = \pi^b(s_t) - \pi^u(s_t) - [V_t^e(0) - W_t^e(0)] + (1-\delta)\beta\mathcal{S}_{t+1}. \tag{41}$$

Solving for ϕ_t gives

$$\phi_t = \lambda^b \nu \psi\left(k_t^b - m_t^b\right) + \lambda^b(1-\nu)\psi\left(\hat{k}\right) + \eta\left[\pi^b(s_t) - \pi^u(s_t)\right] - \eta\left[V_t^e(0) - W_t^e(0)\right].$$

Using that from (6), $V_t^e(0) = \alpha_t X_t^e(0) + (1-\alpha_t)W_t^e(0)$, it follows that $V_t^e(0) - W_t^e(0) = \alpha_t\left[X_t^e(0) - W_t^e(0)\right] = \beta\alpha_t\mathcal{S}_{t+1}^e$. From the bargaining, $\mathcal{S}_{t+1}^e = (1-\eta)\mathcal{S}_{t+1}^b/\eta$ and from the free-entry condition in (7), $\mathcal{S}_{t+1}^b = \zeta\theta_t/(\beta\alpha_t)$. Putting all this together, we obtain (20). ■

Proof of Lemma 1. From (6), $V_t^e(0) - W_t^e(0) = \alpha_t\beta\mathcal{S}_{t+1}^e = \alpha_t(1-\eta)\beta\mathcal{S}_{t+1}$ where we used the generalized Nash solution, i.e. $\mathcal{S}_{t+1}^e = (1-\eta)\mathcal{S}_{t+1}$. Substituting into (41) to obtain:

$$\mathcal{S}_t = \Delta\pi(s_t) - \alpha_t(1-\eta)\beta\mathcal{S}_{t+1} + (1-\delta)\beta\mathcal{S}_{t+1}. \tag{42}$$

The first two terms on the right side of (42), $\Delta\pi(s_t) - \alpha_t(1-\eta)\beta\mathcal{S}_{t+1}$, represent the flow surplus from a lending relationship: it is the increase in the expected profits of the entrepreneur from having access to external finance net of the entrepreneur's outside option. From (7), assuming positive entry in equilibrium, $V_t^b = V_{t+1}^b = 0$ implies $\zeta\theta_t = \alpha_t\eta\beta\mathcal{S}_{t+1}$. Substituting $\mathcal{S}_{t+1} = \zeta\theta_t/\left(\eta\beta\alpha_t\right)$ into (42) gives (25). ■

Proof of Proposition 2. From (25) with $\theta_t = \theta_{t+1} = \theta$, steady-state credit market tightness is the unique solution to

$$(\rho+\delta)\frac{\theta}{\alpha(\theta)} + (1-\eta)\theta = \frac{\eta\left[\pi^b(s) - \pi^u(s)\right]}{\zeta}. \tag{43}$$

Using that the left side is increasing in θ and $\lim_{\theta\to 0}\theta/\alpha(\theta) = 1$, (43) admits a positive solution if

$$(\rho+\delta)\zeta < \eta\left[\pi^b(s) - \pi^u(s)\right].$$

By differentiating (43) and using that $\pi^{b\prime}(s) = -\nu m^b$ and $\pi^{u\prime}(s) = -\nu m^u$ we obtain:

$$\frac{\partial\theta}{\partial s} = \eta\frac{\nu\left(m^u - m^b\right)}{\zeta}\left\{(\rho+\delta)\frac{[1-\epsilon(\theta)]}{\alpha(\theta)} + (1-\eta)\right\}^{-1}. \tag{44}$$

From (15) and (19), if $\lambda^b = \lambda^u = \lambda$, then $m^u > m^b$ for all $s > 0$. Hence, from the expression above, $\partial\theta/\partial s > 0$ for all $s > 0$.

Given θ, closed-form solutions for $(\ell, m^u, m^b, k^b, \phi)$ are obtained recursively as follows:

$$\ell = \frac{\alpha(\theta)}{\delta + \alpha(\theta)} \tag{45}$$

$$\frac{s}{\lambda} = y'(m^u) - 1 \tag{46}$$

$$m^b = \max\left\{m^u - \psi'^{-1}\left(\frac{s}{\lambda}\right), 0\right\} \tag{47}$$

$$k^b = \max\left\{m^u, \hat{k}\right\} \tag{48}$$

$$\phi = \lambda(1-\nu)\psi(\hat{k}) + \lambda\nu\psi(k^b - m^b) + \eta\left[\pi^b(s) - \pi^u(s)\right] - (1-\eta)\zeta\theta. \tag{49}$$

Equation (45) is obtained from (26). Equation (46) corresponds to (15). Equation (47) corresponds to (19) where we used that if $m^b > 0$ then $k^b = k^u = m^u$ and hence $\psi'\left(k^b - m^b\right) = s/\lambda$. By taking the inverse of ψ', $m^b = m^u - \psi'^{-1}(s/\lambda)$. If $m^b = 0$, then $y'(k^b) - 1 = \psi'\left(k^b\right) \leq s/\lambda = y'(m^u) - 1$. It follows that $m^u \leq k^b \leq \psi'^{-1}(s/\lambda)$ where $k^b = \hat{k}$. This gives (48). Finally, (49) is obtained from (20).

The low-spread regime corresponds to the case where the constraint $m^b \geq 0$ does not bind. Hence $k^b = k^u = m^u$ and $m^b = m^u - \psi'^{-1}(s/\lambda)$, which corresponds to (28). The condition $m^b \geq 0$ for all s such that $m^u \geq \psi'^{-1}(s/\lambda)$, i.e., $y'^{-1}(1 + s/\lambda) \geq \psi'^{-1}(s/\lambda)$. The left side is decreasing in s while the right side is increasing in s, so there is a threshold $\hat{s}$ such that the inequality holds for all $s \leq \hat{s}$. The threshold solves $y'^{-1}(1 + \hat{s}/\lambda) = \psi'^{-1}(\hat{s}/\lambda)$, i.e., $y'(\hat{k}) - 1 = \psi'\left(\hat{k}\right) = \hat{s}/\lambda$. The high-spread regime corresponds to the case where the constraint $m^b \geq 0$ binds, in which case $k^b = \hat{k}$ and $s > \hat{s}$.

Finally, in the neighborhood of $s = 0$, from (46) and (47), $m^u = m^b \approx k^*$. From (44), $\partial\theta/\partial s \approx 0$. Differentiating (46) and (47),

$$\frac{\partial m^b}{\partial s} = \frac{1}{\lambda y''(k^*)} - \frac{1}{\lambda\psi''(0)} < \frac{\partial m^u}{\partial s} = \frac{1}{\lambda y''(k^*)} < 0.$$

■

Proof of Proposition 3. From (15) and (19), m^b and m^u are continous functions of s such that $m^b = m^u = k^*$ when $s = 0$. If $\partial\left(m^b - m^u\right)/\partial s > 0$ when evaluated at $s = 0^+$ then there exists $s_0 > 0$ such that $m^b > m^u$ for all $s < s_0$. By differentiating the FOC (19),

$$\frac{\partial m^b}{\partial s} = \frac{1}{y''(k^b)\lambda^b} - \frac{1}{\psi''(k^b - m^b)\lambda^b}.$$

Using that $\psi''(k^b - m^b) = \psi_0(k^b - m^b)^{\xi - 1}/\xi$, in the neighborhood of $s = 0^+$,

$$\frac{\partial m^b}{\partial s} = \begin{cases} \frac{1}{y''(k^*)\lambda^b} & \\ \frac{1}{y''(k^*)\lambda^b} - \frac{1}{\psi_0\lambda^b} & \text{if } \xi \quad \begin{matrix} < 1 \\ = 1 \\ > 1. \end{matrix} \\ -\infty & \end{cases}$$

The condition $\partial m^b/\partial s > \partial m^u/\partial s = 1/\left[y''(k^*)\lambda^u\right]$ holds if $\xi < 1$; it is equivalent to

$$\frac{\lambda^b - \lambda^u}{\lambda^u} > \frac{-y''(k^*)}{\psi_0}$$

if $\xi = 1$; it does not hold if $\xi > 1$. Hence, if $\xi < 1$ or $\xi = 1$ and $\left(\lambda^b - \lambda^u\right)/\lambda^u > -y''(k^*)/\psi_0$, then there exists $s_0 > 0$ such that $m^b > m^u$ for all $s < s_0$. By the same reasoning as in the proof of Proposition 2, for all $s > \lambda^b\psi'\left(\hat{k}\right)$, $m^b = 0 < m^u$. The effect of a change in s on market tightness is obtained from (44) in the proof of Proposition 2, according to which $\partial\theta/\partial s \sim \Delta\pi'(s) = \nu\left(m^u - m^b\right)$. ■

Proposition 4 ***(Implementing constrained efficient allocations.)*** *The equilibrium achieves the constrained-efficient allocation if and only if $s_t = 0$ and $\epsilon(\theta_t) = \eta$ for all t.*

Proof of Proposition 4. We measure social welfare starting in stage 2 of period 0 as the discounted sum of aggregate output flows net of the costs associated with production, intermediation, and bank entry: $\mathbb{W}(\ell_0) = \sum_{t=0}^{\infty}\beta^t\mathcal{W}_t$ where the period welfare is

$$\begin{aligned}\mathcal{W}_t \;=\;& -\zeta\theta_t(1-\ell_t) + \beta(1-\ell_{t+1})\lambda^u\nu\left[y\left(k_{t+1}^u\right) - k_{t+1}^u\right] \\ & +\beta\ell_{t+1}\lambda^b\left\{\nu\left[y(k_{t+1}^b) - k_{t+1}^b - \psi\left(\mathrm{L}_{t+1}\right)\right] + (1-\nu)\left[y(\hat{k}_{t+1}^b) - \hat{k}_{t+1}^b - \psi\left(\hat{\mathrm{L}}_{t+1}\right)\right]\right\}.\end{aligned}\tag{50}$$

The first term on the right side of (50) are banks' entry costs in the relationship lending market; the second term represents the profits of unbanked entrepreneurs; and the third term represents the profits of banked entrepreneurs net of the costs of external finance. The planner chooses $\left\{\theta_t, k_{t+1}^u, k_{t+1}^b, \hat{k}_{t+1}^b, \mathrm{L}_{t+1}, \hat{\mathrm{L}}_{t+1}\right\}_{t=0}^{\infty}$ to maximize $\mathbb{W}(\ell_0)$ subject to the constraint imposed by the matching technology, $\alpha(\theta_t)$, and the financial constraint according to which the investment of entrepreneurs who are victims of theft must be financed with bank loans, $\hat{k}_{t+1}^b \leq \hat{\mathrm{L}}_{t+1}$.

From the maximization of (50) with respect to $(k_{t+1}^u, k_{t+1}^b, \hat{k}_{t+1}^b, \mathrm{L}_{t+1}, \hat{\mathrm{L}}_{t+1})$, any constrained-efficient allocations satisfy:

$$k_{t+1}^u = k_{t+1}^b = k^* \tag{51}$$

$$\mathrm{L}_{t+1} = 0 \tag{52}$$

$$\hat{k}_{t+1}^b = \hat{\mathrm{L}}_{t+1} = \hat{k}. \tag{53}$$

According to (51), the planner chooses the first-best level of investment, k^*, to maximize profits whenever it is not subject to the bank financing constraint. In that case, from (52), the loan size is zero. From (53), whenever a bank loan is the only option to finance an investment opportunity, then the optimal investment is $\hat{k} = \arg\max\left[y(k) - k - \psi(k)\right]$. The comparison of the equilibrium conditions (15) and (19) with (51) shows that a necessary condition for the implementation of a constrained-efficient allocation is $s_{t+1} = 0$ for all t.

Using (51)-(53), we write the planner's problem recursively as

$$\begin{aligned}\mathbb{W}(\ell_t) \;=\;& \max_{\theta_t\geq 0}\left\{-\zeta\theta_t(1-\ell_t) + \beta(1-\ell_{t+1})\lambda^u\nu\left[y\left(k^*\right) - k^*\right]\right. \\ & \left.+\beta\ell_{t+1}\lambda^b\left\{\nu\left[y(k^*) - k^*\right] + (1-\nu)\left[y(\hat{k}) - \hat{k} - \psi\left(\hat{k}\right)\right]\right\} + \beta\mathbb{W}(\ell_{t+1})\right\},\end{aligned}\tag{54}$$

where $\ell_{t+1} = (1-\delta)\ell_t + \alpha(\theta_t)(1-\ell_t)$. Assuming an interior solution, the planner's optimality conditions are:

$$\zeta = \alpha'(\theta_t)\beta\omega_t \tag{55}$$

$$\begin{aligned}\omega_t \;=\;& \left(\lambda^b - \lambda^u\right)\nu\left[y(k^*) - k^*\right] + \lambda^b(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi\left(\hat{k}\right)\right] \\ & +\beta\left\{1 - \delta - \alpha(\theta_{t+1})\left[1 - \epsilon(\theta_{t+1})\right]\right\}\omega_{t+1},\end{aligned}\tag{56}$$

where $\omega_t = (1+\rho)\left[\mathbb{W}'(\ell_t) - \zeta\theta_t\right] / \left[1-\delta-\alpha(\theta_t)\right]$. From (56), one can interpret ω_t as the social value of a lending relationship. According to (55), the planner equalizes the cost of bank entry with the product of the marginal rate of relation creation, $\alpha'(\theta_t)$, and the discounted value of a relationship, $\beta\omega_t$. From (25), in equilibrium the free-entry condition for banks when $s_{t+1} = 0$ is

$$\zeta = \frac{\alpha(\theta_t)}{\theta_t}\beta\eta\mathcal{S}_{t+1},$$

where

$$\mathcal{S}_{t+1} = \left(\lambda^b - \lambda^u\right)\nu\left[y(k^*) - k^*\right] + \lambda^b(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi\left(\hat{k}\right)\right] + \beta\left[(1-\delta) - \alpha(\theta_{t+1})(1-\eta)\right]\mathcal{S}_{t+2}.$$

This equilibrium condition coincides with the planner's optimality conditions, (55) and (56), if and only if $\epsilon(\theta_t) = \eta$. ■

Proposition 5 ***(Suboptimality of the Friedman rule.)*** *Suppose* $\lambda^u = \lambda^b = \lambda$, $\psi(\mathrm{L}) = \mathrm{L}^{1+\xi}/(1+\xi)$ *with* $\xi > 1$, *and*

$$\zeta < \frac{\alpha'(0)\eta\lambda(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi(\hat{k})\right]}{\rho+\delta}. \tag{57}$$

It is optimal to deviate from $s_t \equiv 0$ *if*

$$\frac{\epsilon(\underline{\theta}) - \eta}{1-\epsilon(\underline{\theta})} > \left[\frac{(1-\delta)\ell_0}{\alpha(\underline{\theta})\,(1-\ell_0)} + 1\right]\frac{1}{\xi}, \tag{58}$$

where $\underline{\theta}$ *is steady-state credit market tightness at the Friedman rule.*

Proof of Proposition 5. The economy starts with ℓ_0 lending relationships. We measure social welfare in the second stage of $t = 0$, before banks make entry decisions and entrepreneurs make portfolio decisions, by $\mathbb{W}_0 = \sum_{t=1}^{\infty}\beta^t\mathcal{W}_t$ where

$$\begin{aligned}\mathcal{W}_t \quad = \quad & -(1+\rho)\zeta(1-\ell_{t-1})\theta_{t-1} + (1-\ell_t)\lambda\nu\left[y\left(k_t^u\right) - k_t^u\right] \\ & + \ell_t\lambda\left\{\nu\left[y(k_t^b) - k_t^b - \psi\left(k_t^b - m_t^b\right)\right] + (1-\nu)\left[y(\hat{k}) - \hat{k} - \psi\left(\hat{k}\right)\right]\right\}.\end{aligned} \tag{59}$$

The first term on the RHS is the entry cost of banks in period $t-1$ where $(1-\ell_{t-1})\theta_{t-1}$ is the measure of banks entering. The following terms are the entrepreneurs' profits net of banks' monitoring costs in period t. (Relative to the main text, $\mathcal{W}_t$ has been scaled up by $(1+\rho)$.)

We consider a small deviation of the interest rate spread from $s_1 = 0$. For $t \geq 2$, $s_t = 0$. As a result, for all $t \geq 1$, $\theta_t = \underline{\theta}$ solution to

$$(\rho+\delta)\,\frac{\underline{\theta}}{\alpha(\underline{\theta})} + (1-\eta)\underline{\theta} = \frac{\eta\lambda(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi(\hat{k})\right]}{\zeta} \tag{60}$$

where we used that $\pi^b(0) = \lambda(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi(\hat{k})\right] + \lambda\nu\left[y(k^*) - k^*\right]$ and $\pi^u(0) = \lambda\nu\left[y(k^*) - k^*\right]$. From (57), $\underline{\theta} > 0$. For all $t \geq 2$, $m_t^u = m_t^b = k^*$. From (26), the measure of lending relationships solves:

$$\begin{aligned}\ell_1 \quad &= \quad (1-\delta)\ell_0 + \alpha(\theta_0)(1-\ell_0) && (61)\\ \ell_t \quad &= \quad \underline{\ell} + (\ell_1 - \underline{\ell})\left[1-\delta-\alpha(\underline{\theta})\right]^{t-1} \quad \text{for all } t \geq 1, && (62)\end{aligned}$$

where we have used that ℓ_t is the solution to the following linear, first-order difference equation, $\ell_{t+1} = (1-\delta)\ell_t + \alpha(\underline{\theta})(1-\ell_t)$, with initial condition ℓ_1. The long-run solution is $\underline{\ell} = \alpha(\underline{\theta})/\left[\delta + \alpha(\underline{\theta})\right]$. The welfare starting in the second stage of $t = 1$ is measured by $\mathbb{W}_1^0 = \sum_{t=2}^{\infty} \beta^{t-2} \mathcal{W}_t^0$ where

$$\mathcal{W}_t^0 = -(1+\rho)\zeta(1-\ell_{t-1})\underline{\theta} + \lambda\nu\left[y\left(k^*\right) - k^*\right] + \ell_t \lambda(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi\left(\hat{k}\right)\right],$$

where we have used that $\theta_{t-1} = \underline{\theta}$ and $m_t^u = m_t^b = k^*$. First, we rearrange the terms of the sum to rewrite $\mathbb{W}_1^0$ as follows:

$$\begin{aligned}\mathbb{W}_1^0 &= (1+\rho)\zeta\ell_1\underline{\theta} + \sum_{t=2}^{\infty} \beta^{t-2}\left\{-(1+\rho)\zeta\underline{\theta} + \lambda\nu\left[y\left(k^*\right) - k^*\right]\right\} \\ &\quad + \sum_{t=2}^{\infty} \beta^{t-2}\ell_t\left\{\zeta\underline{\theta} + \lambda(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi\left(\hat{k}\right)\right]\right\}. \end{aligned} \tag{63}$$

The second term on the right side of (63) is equal to:

$$\sum_{t=2}^{\infty} \beta^{t-2}\left\{-(1+\rho)\zeta\underline{\theta} + \lambda\nu\left[y\left(k^*\right) - k^*\right]\right\} = \frac{-(1+\rho)\zeta\underline{\theta} + \lambda\nu\left[y\left(k^*\right) - k^*\right]}{1-\beta}. \tag{64}$$

Using that

$$\begin{aligned}\sum_{t=2}^{\infty} \beta^{t-2}\ell_t &= \sum_{t=2}^{\infty} \beta^{t-2}\left\{\underline{\ell} + (\ell_1 - \underline{\ell})[1-\delta-\alpha(\underline{\theta})]^{t-1}\right\} \\ &= \frac{\underline{\ell}}{1-\beta} + \frac{(\ell_1 - \underline{\ell})[1-\delta-\alpha(\underline{\theta})]}{1-\beta[1-\delta-\alpha(\underline{\theta})]},\end{aligned}$$

the third term on the right side of (63) is equal to:

$$\begin{aligned}&\sum_{t=2}^{\infty} \beta^{t-2}\ell_t\left\{\zeta\underline{\theta} + \lambda(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi\left(\hat{k}\right)\right]\right\} = \\ &\left\{\zeta\underline{\theta} + \lambda(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi\left(\hat{k}\right)\right]\right\}\left(\frac{\underline{\ell}}{1-\beta} + \frac{(\ell_1 - \underline{\ell})[1-\delta-\alpha(\underline{\theta})]}{1-\beta[1-\delta-\alpha(\underline{\theta})]}\right).\end{aligned} \tag{65}$$

Substituting (64) and (65) into (63), and after some calculation:

$$\begin{aligned}\mathbb{W}_1^0 &= \frac{-(1-\underline{\ell})(1+\rho)\zeta\underline{\theta} + \lambda\nu\left[y\left(k^*\right) - k^*\right] + \underline{\ell}\lambda(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi\left(\hat{k}\right)\right]}{1-\beta} \\ &\quad + \frac{(1+\rho)\zeta\underline{\theta} + [1-\delta-\alpha(\underline{\theta})]\lambda(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi\left(\hat{k}\right)\right]}{1-\beta[1-\delta-\alpha(\underline{\theta})]}(\ell_1 - \underline{\ell}).\end{aligned} \tag{66}$$

We are now in position to measure welfare from $t=0$:

$$
\begin{aligned}
(1+\rho)\mathbb{W}_0 &= \mathcal{W}_1+\beta\mathbb{W}_1^0 \\
&= -(1+\rho)\zeta(1-\ell_0)\theta_0+(1-\ell_1)\lambda\nu\left[y\left(k_1^u\right)-k_1^u\right] \\
&\quad +\ell_1\lambda\left\{\nu\left[y(k_1^b)-k_1^b-\psi\left(k_1^b-m_1^b\right)\right]+(1-\nu)\left[y(\hat{k})-\hat{k}-\psi\left(\hat{k}\right)\right]\right\} \\
&\quad +\frac{\zeta\underline{\theta}+\beta[1-\delta-\alpha(\underline{\theta})]\lambda(1-\nu)\left[y(\hat{k})-\hat{k}-\psi\left(\hat{k}\right)\right]}{1-\beta[1-\delta-\alpha(\underline{\theta})]}\ell_1 \\
&\quad -\frac{\zeta\underline{\theta}+\beta[1-\delta-\alpha(\underline{\theta})]\lambda(1-\nu)\left[y(\hat{k})-\hat{k}-\psi\left(\hat{k}\right)\right]}{1-\beta[1-\delta-\alpha(\underline{\theta})]}\underline{\ell} \\
&\quad +\frac{-(1-\underline{\ell})\zeta\underline{\theta}+\beta\lambda\nu\left[y\left(k^*\right)-k^*\right]+\beta\underline{\ell}\lambda(1-\nu)\left[y(\hat{k})-\hat{k}-\psi\left(\hat{k}\right)\right]}{1-\beta},
\end{aligned} \tag{67}
$$

where we obtained the second equality by substituting $\mathbb{W}_1^0$ by its expression given by (66) and we used (59) to obtain

$$
\begin{aligned}
\mathcal{W}_1 &= -(1+\rho)\zeta(1-\ell_0)\theta_0+(1-\ell_1)\lambda\nu\left[y\left(k_1^u\right)-k_1^u\right] \\
&\quad +\ell_1\lambda\left\{\nu\left[y(k_1^b)-k_1^b-\psi\left(k_1^b-m_1^b\right)\right]+(1-\nu)\left[y(\hat{k})-\hat{k}-\psi\left(\hat{k}\right)\right]\right\}.
\end{aligned}
$$

We differentiate $\tilde{\mathbb{W}}_0=(1+\rho)\mathbb{W}_0$ given by (67) with respect to s_1:

$$
\begin{aligned}
\frac{\partial\tilde{\mathbb{W}}_0}{\partial s_1} &= \frac{\partial\tilde{\mathbb{W}}_0}{\partial\theta_0}\frac{\partial\theta_0}{\partial s_1}+(1-\ell_1)\lambda\nu\left[y'\left(k_1^u\right)-1\right]\frac{\partial k_1^u}{\partial s_1} \\
&\quad +\ell_1\lambda\nu\psi'\left(k_1^b-m_1^b\right)\frac{\partial m_1^b}{\partial s_1},
\end{aligned} \tag{68}
$$

where we used that $\partial\tilde{\mathbb{W}}_0/\partial k_1^u=(1-\ell_1)\lambda\nu\left[y'\left(k_1^u\right)-1\right]$, $\partial\tilde{\mathbb{W}}_0/\partial m_1^b=\ell_1\lambda\nu\psi'\left(k_1^b-m_1^b\right)$. The derivative on the right side of (68) is equal to

$$
\begin{aligned}
(1-\ell_0)^{-1}\frac{\partial\tilde{\mathbb{W}}_0}{\partial\theta_0} &= -(1+\rho)\zeta-\alpha'(\theta_0)\lambda\nu\left[y\left(k_1^u\right)-k_1^u\right] \\
&\quad +\alpha'(\theta_0)\lambda\left\{\nu\left[y(k_1^b)-k_1^b-\psi\left(k_1^b-m_1^b\right)\right]+(1-\nu)\left[y(\hat{k})-\hat{k}-\psi\left(\hat{k}\right)\right]\right\} \\
&\quad +\alpha'(\theta_0)\frac{\zeta\underline{\theta}+\beta[1-\delta-\alpha(\underline{\theta})]\lambda(1-\nu)\left[y(\hat{k})-\hat{k}-\psi\left(\hat{k}\right)\right]}{1-\beta[1-\delta-\alpha(\underline{\theta})]},
\end{aligned}
$$

where we used, from (61), $\partial\ell_1/\partial\theta_0=\alpha'(\theta_0)(1-\ell_0)$. From (25) credit market tightness at $t=0$ solves:

$$
\frac{\theta_0}{\alpha(\theta_0)}=\frac{\beta\eta\left[\Delta\pi(s_1)\right]}{\zeta}-\beta(1-\eta)\underline{\theta}+\beta(1-\delta)\frac{\underline{\theta}}{\alpha(\underline{\theta})}. \tag{69}
$$

By differentiating (69) and using that $\partial\Delta\pi(s_1)/\partial s_1=\nu\left(m_1^u-m_1^b\right)$, we obtain:

$$
\frac{\partial\theta_0}{\partial s_1}=\frac{\alpha(\theta_0)}{1-\epsilon(\theta_0)}\frac{\beta\eta\nu(m_1^u-m_1^b)}{\zeta}. \tag{70}
$$

Differentiating (15) and (19) assuming $m_1^b > 0$,

$$\frac{\partial m_1^u}{\partial s_1} = \frac{1}{\lambda y''(m_1^u)} \tag{71}$$

$$\frac{\partial m_1^b}{\partial s_1} = \frac{\psi''\left(m_1^u - m_1^b\right) - y''(m_1^u)}{\psi''\left(m_1^u - m_1^b\right)\lambda y''(m_1^u)}. \tag{72}$$

Substituting (70), (71), and (72) into $\partial \mathbb{W}_0/\partial s_1$ and rearranging, we obtain:

$$\begin{aligned}\frac{\partial \tilde{\mathbb{W}}_0}{\partial s_1} &= \frac{\partial \tilde{\mathbb{W}}_0}{\partial \theta_0}\frac{\alpha(\theta_0)}{1-\epsilon(\theta_0)}\frac{\beta\eta\nu(m_1^u - m_1^b)}{\zeta} \\ &+(1-\ell_1)\frac{\nu\psi'\left(m_1^u - m_1^b\right)}{y''(m_1^u)} \\ &+\ell_1\nu\psi'\left(m_1^u - m_1^b\right)\frac{\psi''\left(m_1^u - m_1^b\right) - y''(m_1^u)}{\psi''\left(m_1^u - m_1^b\right)y''(m_1^u)},\end{aligned}$$

where we used that for s_1 small enough, $k_1^b = k_1^u = m_1^u$, and, from (19), $\psi'\left(k_1^b - m_1^b\right) = y'(k_1^b) - 1$. Dividing by $\nu\left(m_1^u - m_1^b\right)$ and taking the limit as s_1 approaches 0:

$$\begin{aligned}\lim_{s_1\to 0}\left(\frac{1}{\nu\left(m_1^u - m_1^b\right)}\frac{\partial \tilde{\mathbb{W}}_0}{\partial s_1}\right) &= \frac{\partial \tilde{\mathbb{W}}_0}{\partial \theta_0}\frac{\alpha(\underline{\theta})}{1-\epsilon(\underline{\theta})}\frac{\beta\eta}{\zeta} \\ &+(1-\ell_1)\psi''(0)\frac{1}{y''(k^*)} \\ &+\ell_1\frac{\psi''(0) - y''(k^*)/\xi(0)}{y''(k^*)},\end{aligned} \tag{73}$$

where

$$(1-\ell_0)^{-1}\left.\frac{\partial \tilde{\mathbb{W}}_0}{\partial \theta_0}\right|_{s_1=0} = -(1+\rho)\zeta + \alpha'(\underline{\theta})\frac{\lambda(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi\left(\hat{k}\right)\right] + \zeta\underline{\theta}}{1-\beta[1-\delta-\alpha(\underline{\theta})]}. \tag{74}$$

From the free-entry condition (69) when $s_1 = 0$:

$$\frac{(1+\rho)\underline{\theta}\zeta}{\eta\alpha(\underline{\theta})} = \frac{\lambda(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi\left(\hat{k}\right)\right] + \zeta\underline{\theta}}{1-\beta[1-\delta-\alpha(\underline{\theta})]}, \tag{75}$$

where we used that $\theta_0 = \underline{\theta}$ and $\Delta\pi(0) = \lambda(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi\left(\hat{k}\right)\right]$. Substituting (75) into (74):

$$(1-\ell_0)^{-1}\left.\frac{\partial \tilde{\mathbb{W}}_0}{\partial \theta_0}\right|_{s_1=0} = (1+\rho)\zeta\left[\frac{\epsilon(\underline{\theta}) - \eta}{\eta}\right].$$

Substituting this expression into (73), $s_1 > 0$ is optimal if:

$$\epsilon(\underline{\theta}) - \eta > \frac{1-\epsilon(\underline{\theta})}{\alpha(\underline{\theta})}\left[-\left(\frac{1-\ell_1}{1-\ell_0}\right)\frac{\psi''(0)}{y''(k^*)} - \frac{\ell_1}{1-\ell_0}\frac{\psi''(0) - y''(k^*)/\xi(0)}{y''(k^*)}\right],$$

where, from (61), $\ell_1 = (1-\delta)\ell_0 + \alpha(\underline{\theta})(1-\ell_0)$. Assume $\psi(\text{L}) = \text{L}^{1+\xi}/(1+\xi)$ with $\xi > 1$. The condition above can be rewritten as:

$$\epsilon(\underline{\theta}) - \eta > \frac{1-\epsilon(\underline{\theta})}{\alpha(\underline{\theta})}\left(\frac{\ell_1}{1-\ell_0}\frac{1}{\xi}\right).$$

Plugging ℓ_1 by its expression we obtain (58). ■

Proposition 6 ***(Ramsey problem.)*** *The policymaker's value function solves*

$$\mathbb{W}(\ell_0) = \max_{\theta_0 \in \Omega = [\underline{\theta}, \bar{\theta}]} \widetilde{\mathbb{W}}(\ell_0, \theta_0), \tag{76}$$

where $\widetilde{\mathbb{W}}$ *is the unique solution in* $\mathcal{B}([0,1] \times \Omega)$ *to (29) where*

$$\underline{\theta} = \frac{\frac{\bar{\alpha}\beta\eta}{\zeta}\left\{\left(\lambda^b - \lambda^u\right)\nu\left[y(k^*) - k^*\right] + \lambda^b(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi(\hat{k})\right]\right\} + \beta(1-\delta) - 1}{1 - \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]} \tag{77}$$

$$\bar{\theta} = \frac{\bar{\alpha}\beta\eta\lambda^b\left[y(\hat{k}) - \hat{k} - \psi(\hat{k})\right]/\zeta + \beta(1-\delta) - 1}{1 - \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]}. \tag{78}$$

Proof of Proposition 6. First, we characterize the state space. Given the functional form $\alpha(\theta) = \bar{\alpha}\theta/(1+\theta)$ and the parametric condition $\delta + \bar{\alpha}(1-\eta) < 1$, the law of motion for market tightness, (25), can be rewritten as

$$\theta_t = \frac{\bar{\alpha}\beta\eta}{\zeta}\Delta\pi(s_{t+1}) + \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]\theta_{t+1} + \beta(1-\delta) - 1. \tag{79}$$

We restrict the policymaker's choice to bounded sequences $\{\theta_t\}$ that solve (79) given some initial condition, θ_0. The set of values for market tightness, Ω, is obtained as follows. We define $\bar{\theta}$ as steady-state credit market tightness when $s = \infty$, in which case $\Delta\pi(+\infty) = \lambda^b\left[y(\hat{k}) - \hat{k} - \psi(\hat{k})\right]$. From (79) it solves:

$$\bar{\theta} = \frac{\bar{\alpha}\beta\eta}{\zeta}\Delta\pi(+\infty) + \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]\bar{\theta} + \beta(1-\delta) - 1.$$

Solving for $\bar{\theta}$, we obtain (78), i.e.,

$$\bar{\theta} = \frac{\bar{\alpha}\beta\eta\lambda^b\left[y(\hat{k}) - \hat{k} - \psi(\hat{k})\right]/\zeta + \beta(1-\delta) - 1}{1 - \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]}.$$

Suppose $\theta_t > \bar{\theta}$ for some t. From (79),

$$\theta_{t+1} - \bar{\theta} = \frac{\theta_t - \bar{\theta} - \bar{\alpha}\beta\eta\left\{\Delta\pi(s_{t+1}) - \Delta\pi(+\infty)\right\}/\zeta}{\beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]}.$$

For all $s_{t+1} \in [0, \infty)$, $\Delta\pi(s_{t+1}) - \Delta\pi(\infty) \leq 0$. Since $\beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right] \in (0,1)$, the sequence $\{\theta_t - \bar{\theta}\}$ is increasing and unbounded, which is inconsistent with an equilibrium. Next, we define $\underline{\theta}$ as steady-state market tightness when $s = 0$, in which case $\Delta\pi(0) = \left(\lambda^b - \lambda^u\right)\nu\left[y(k^*) - k^*\right] + \lambda^b(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi(\hat{k})\right]$. From (79), it solves:

$$\underline{\theta} = \frac{\bar{\alpha}\beta\eta}{\zeta}\Delta\pi(0) + \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]\underline{\theta} + \beta(1-\delta) - 1.$$

Solving for $\underline{\theta}$, we obtain (77), i.e.,

$$\underline{\theta} = \frac{\bar{\alpha}\beta\eta\left\{\left(\lambda^b - \lambda^u\right)\nu\left[y(k^*) - k^*\right] + \lambda^b(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi(\hat{k})\right]\right\}/\zeta + \beta(1-\delta) - 1}{1 - \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]}.$$

Suppose $\theta_t \in (0, \underline{\theta})$ for some t. With $\underline{\theta} > 0$, from (79),

$$\theta_{t+1} - \underline{\theta} = \frac{\theta_t - \underline{\theta} - \bar{\alpha}\beta\eta\left\{\Delta\pi(s_{t+1}) - \Delta\pi(0)\right\}/\zeta}{\beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]}.$$

For all $s_{t+1} \in [0, +\infty)$, $\Delta\pi(s_{t+1}) - \Delta\pi(0) \geq 0$. So θ_t becomes negative in finite time, which is inconsistent with an equilibrium. Finally, for all $\theta_t \in [\underline{\theta}, \bar{\theta}]$ there exists a θ_{t+1} consistent with an equilibrium, e.g., a steady-state path where $\theta_{t+\tau} = \theta_t$ for all $\tau > 1$.

The feasibility condition $\theta_{t+1} \in \Gamma(\theta_t)$ is obtained from (79) by varying s_{t+1} from 0 to $+\infty$, i.e.,

$$\Gamma(\theta_t) = \left[\frac{\theta_t - \bar{\alpha}\beta\eta\left[\left(\lambda^b - \lambda^u\right)\nu\left[y(k^*) - k^*\right] + \lambda^b(1-\nu)\left[y(\hat{k}) - \hat{k} - \psi(\hat{k})\right]\right]/\zeta - \beta(1-\delta) + 1}{\beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]},\right.$$
$$\left.\frac{\theta_t - \bar{\alpha}\beta\eta\lambda^b\left[y(\hat{k}) - \hat{k} - \psi(\hat{k})\right]/\zeta - \beta(1-\delta) + 1}{\beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]}\right] \cap \Omega. \tag{80}$$

Given a $\theta_{t+1} \in \Gamma(\theta_t)$, the remaining choice variables of the planner are determined recursively according to:

$$\Delta\pi(s_{t+1}) = \frac{\zeta\left\{\theta_t - \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]\theta_{t+1} - \beta(1-\delta) + 1\right\}}{\bar{\alpha}\beta\eta} \tag{81}$$

$$m^u_{t+1} = y'^{-1}\left(1 + \frac{s_{t+1}}{\lambda^u}\right) \tag{82}$$

$$k^b_{t+1} = \max\left\{y'^{-1}\left(1 + \frac{s_{t+1}}{\lambda^b}\right), \hat{k}\right\} \tag{83}$$

$$\mathrm{L}_{t+1} = \psi'^{-1}\left[y'(k^b_{t+1}) - 1\right] \tag{84}$$

$$\ell_{t+1} = (1-\delta)\ell_t + \frac{\bar{\alpha}\theta_t}{1+\theta_t}(1-\ell_t). \tag{85}$$

We now turn to the Bellman equation (29). For a given initial market tightness, θ_0, we can apply the Principle of Optimality to show the value function of the planner, $\widetilde{\mathbb{W}}(\ell_t, \theta_t)$, solves the Bellman equation (29). It is the fixed point of a mapping from $\mathcal{B}([0,1] \times [\underline{\theta}, \bar{\theta}])$ into itself. The mapping in (29) is a contraction by Blackwell's sufficient conditions (Theorem 3.3 in Stokey and Lucas, 1989), and by the contraction mapping theorem (Theorem 3.2 in Stokey and Lucas 1989), the fixed point exists and is unique. The correspondence Γ is continuous and the policymaker's period utility is also continuous. So $\widetilde{\mathbb{W}}(\ell, \theta)$ is continuous by the Contraction Mapping Theorem. Given there is no initial value for θ in the original sequence problem, (30), $\theta_0 \in \Omega$ is chosen to as to maximize $\widetilde{\mathbb{W}}(\ell_0, \theta_0)$. Such a solution exists by the continuity of $\widetilde{\mathbb{W}}(\ell_0, \theta_0)$ and the compactness of Ω. ■

Appendix A2: Endogenous theft

In our main model, the probability that the liquid assets of the entrepreneurs are embezzled, $1-\nu$, is exogenous. To the extent that monetary policy affects the quantity of liquid assets held by entrepreneurs, it could also affect the incentives of corporate thieves to engage in embezzlement activities.

In order to endogenize ν, we assume that there is a unit measure of corporate thieves. Corporate thieves can be workers, managers, or accountants, or any other person who can access the liquid assets of the entrepreneur. At the start of each period, thieves exert some effort, $e \in [0,1]$, that will allow them to embezzle the liquid assets of the entrepreneur they will be matched with. For instance, they learn fraud techniques and methods to divert funds without being caught. The cost associated with this effort is $\varphi(e)$ with $\varphi(0)=\varphi'(0)=0$, $\varphi'(e)>0$, $\varphi''(e)>0$. In stage 1, corporate thieves are matched at random to entrepreneurs. A thief is successful in embezzling the liquid assets of the entrepreneur with probability e. Thieves are risk neutral, i.e., their utility function is $U(c)=c$.

The problem of the corporate thief within a period is:

$$\max_{e\in[0,1]} \left\{-\varphi(e)+e\mathrm{M}_t\right\}, \tag{86}$$

where $\mathrm{M}_t=\ell_t m_t^b+(1-\ell_t)m_t^u$ is aggregate liquidity. The expected value of embezzlement corresponds to the second term in the maximization problem. If the thief is matched to an unbanked entrepreneur, with probability $1-\ell_t$, then he can steal m_t^u with probability e. If he is matched with a banked entrepreneur, with probability ℓ_t, he can steal m_t^b. Assuming an interior solution, the first-order condition gives:

$$\varphi'(e)=\mathrm{M}_t. \tag{87}$$

The marginal cost of embezzlement on the left side is equal to the discounted aggregate liquidity on the right side. Given that the problem is strictly concave, all thieves choose the same e^*.

The probability that an entrepreneur is subject to theft is $1-\nu=e^*$. Hence,

$$\nu=1-\varphi'^{-1}\left(\mathrm{M}_t\right). \tag{88}$$

From the first-order conditions (15) and (19) m_t^b and m_t^u are independent of ν and decreasing with s. So, if $m_t^b<m_t^u$, e.g., if $\lambda^b=\lambda^u=\lambda$, then M_t is decreasing with θ and decreasing with s. It follows from (88) that $\nu=1-e^*$ is increasing with θ and s. At the Friedman rule, $s=0$, $\nu=1-\varphi'^{-1}\left(k^*\right)$.

We now turn to the free-entry condition of banks and the relationship it implies between θ and $\nu=1-e^*$. Note first that

$$\begin{aligned}\frac{\partial\Delta\pi(s)}{\partial\nu} &= -\lambda^b\left[y(\hat{k}^b)-\hat{k}^b-\psi(\hat{k}^b)\right]\\ &\quad+\lambda^b\left[y(k^b)-k^b-\psi(\mathrm{L})\right]-sm^b\\ &\quad-\left\{\lambda^u\left[y(k^u)-k^u\right]-sm^u\right\}\end{aligned}$$

Suppose $\lambda^b=\lambda^u=\lambda$. If s is close to 0, then the last two terms on the right side are approximately equal. Hence, $\partial\Delta\pi(s)/\partial\nu<0$, which implies $\partial\theta/\partial\nu<0$. As the probability of theft is reduced, the benefits from external finance are reduced as well, and hence fewer banks enter. So, from the condition on bank entry, θ is increasing with e^* and s.

We have two equilibrium conditions to determine jointly the pair (θ,e^*). From (87), e^* decreases with θ; from bank entry, θ increases with e^*. So the pair (θ,e^*) is uniquely determined. As s

increases, the curve representing (87) moves to the left in the space (e^*, θ) while the free-entry condition moves up. So, an increase in s reduces e^*.

Figure 10 illustrates the optimal policy response in the calibrated model to a 60% destruction of lending relationships when the policymaker can commitment (top three panels) and when they cannot (bottom three panels). We set $\varphi(M) = \kappa e^2$ and calibrate κ such that ν matches its value in Table 5.2 when the annual spread is 2%. Otherwise the parameters are kept the same as in Table 5.2. The figure illustrates the optimal policy when theft is either exogenous (in solid blue) or when it is endogenous (in dashed-red).

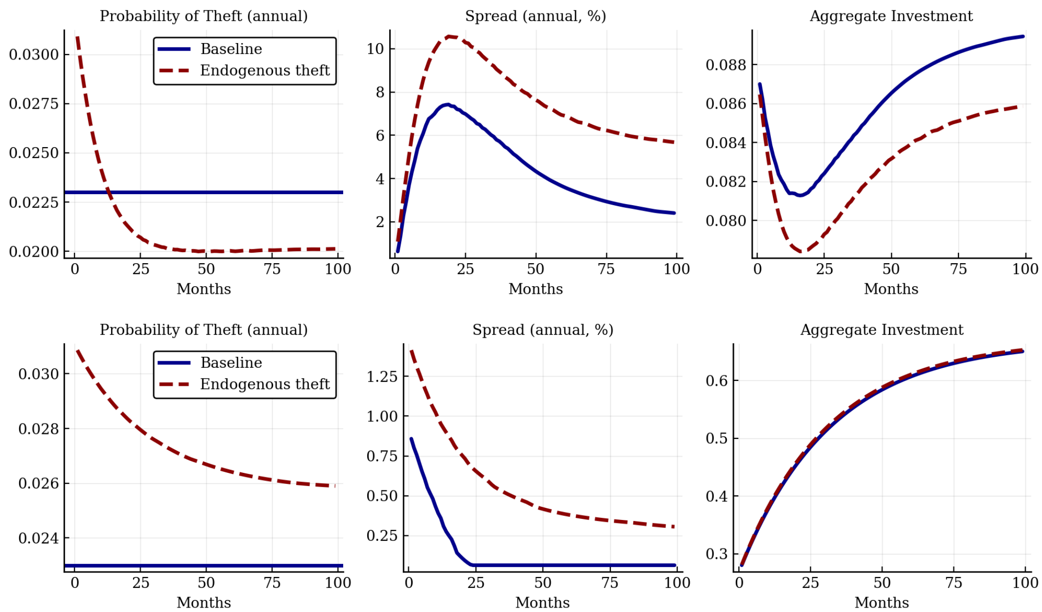

Figure 10: Optimal policy responses with commitment (top three panels) and without commitment (bottom three panels) with endogenous theft.

Qualitatively, the introduction of endogenous theft changes little about the optimal policy prescription. Quantitatively, we see that now the policymaker is introduced to a new tradeoff at the onset of the shock. When the policymaker has commitment, they still lower spreads initially and use forward guidance about future spreads to encourage the creation of lending relationships. However by lowering spreads initially, the policy incentivizes theft so ν increases above its long-run value. The increase in theft both dampens the ability of policy to improve self-insurance and increases the usefulness of bank lending. As a result, spreads are kept low for a shorter amount of time and the long-run value is larger. Without commitment, the initial destruction of relationships causes aggregate liquidity to rise which incentivizes more theft. To counteract the increase in theft, the policymaker increases spreads by more than when ν is fixed and more gradually reduces spreads back to a higher long-run value.

Appendix A3: Continuous time limit

We now derive the continuous-time limit of our model (see Choi and Rocheteau 2020 for a detailed presentation of New Monetarist models in continuous time). We assume there is no theft and the arrival rate of investment opportunities is the same for banked and unbanked entrepreneurs. Let Δ denote the length of a period of time, where Δ is assumed to be small. We rewrite all variables that have a time dimension as being proportional to Δ. It includes the matching function, the separation rate, the rate of time preference, and the real return on liquid assets.

The law of motion of the lending relationships is

$$\ell_{t+\Delta} = (1-\delta\Delta)\ell_t + \alpha(\theta_t)\Delta(1-\ell_t).$$

We subtract ℓ_t on both sides, divide by Δ, and take the limit as Δ goes to zero to obtain

$$\dot{\ell}_t = \alpha(\theta_t)(1-\ell_t) - \delta\ell_t,$$

where $\dot{\ell}_t = \lim_{\Delta\to 0}\left(\ell_{t+\Delta} - \ell_t\right)/\Delta$.

The profits of an unbanked entrepreneur are $\pi_t^u\Delta$ where

$$\pi_t^u = \max_{m_t\geq 0}\left\{-s_t m_t + \lambda^u \max_{k_t\leq m_t}\left[y(k_t) - k_t\right]\right\},$$

where the interest spread is

$$s_t\Delta = \frac{\left(\rho - r_t\right)\Delta}{1+r_t\Delta}.$$

Note that the profits conditional on an investment opportunity, $y(k_t)-k_t$, is a stock that has no time dimension. Taking the limit as Δ goes to zero,

$$s_t = \rho - r_t.$$

The first-order condition gives

$$s_t = \lambda^u\left[y'(m_t^u) - 1\right].$$

The flow profits of banked entrepreneurs are unaffected,

$$\pi^b(s_t) = \max_{k^b,m^b\geq 0}\left\{\lambda^b\left[y(k^b) - k^b - \psi(k^b - m^b)\right] - s_t m^b\right\}.$$

The first-order conditions are

$$\psi'\left(k_t^b - m_t^b\right) = y'(k_t^b) - 1 \leq \frac{s_t}{\lambda^b}, \quad \text{"="} \text{ if } m_t^b > 0, \quad \forall t.$$

Finally, the free-entry condition for banks is

$$(1+\rho\Delta)\frac{\theta_t}{\alpha(\theta_t)\Delta} = \frac{\eta\left[\pi^b(s_{t+\Delta}) - \pi^u\left(s_{t+\Delta}\right)\right]\Delta}{\zeta\Delta} - (1-\eta)\theta_{t+\Delta} + (1-\delta\Delta)\frac{\theta_{t+\Delta}}{\alpha(\theta_{t+\Delta})\Delta}.$$

Notice the cost of bank entry is a flow cost, and hence proportional to Δ. Rearranging, the equation can be rewritten as:

$$\frac{1}{\Delta}\left(\frac{\theta_t}{\alpha(\theta_t)} - \frac{\theta_{t+\Delta}}{\alpha(\theta_{t+\Delta})}\right) + \rho\frac{\theta_t}{\alpha(\theta_t)} + \delta\frac{\theta_{t+\Delta}}{\alpha(\theta_{t+\Delta})} = \frac{\eta\left[\pi^b(s_{t+\Delta}) - \pi^u\left(s_{t+\Delta}\right)\right]}{\zeta} - (1-\eta)\theta_{t+\Delta}.$$

Taking the limit as Δ goes to zero,

$$-\left(\frac{1-\epsilon(\theta_t)}{\alpha(\theta_t)}\right)\dot{\theta}+(\rho+\delta)\,\frac{\theta_t}{\alpha(\theta_t)}=\frac{\eta\left[\pi^b(s_t)-\pi^u\left(s_t\right)\right]}{\zeta}-(1-\eta)\theta_t,$$

where $\epsilon(\theta)\equiv\theta\alpha'(\theta)/\alpha(\theta)$. Rearranging, we obtain:

$$\dot{\theta}_t=\frac{\alpha(\theta_t)}{1-\epsilon(\theta_t)}\left\{(\rho+\delta)\,\frac{\theta_t}{\alpha(\theta_t)}+(1-\eta)\theta_t-\frac{\eta\left[\pi^b(s_t)-\pi^u\left(s_t\right)\right]}{\zeta}\right\}.$$

To summarize, the system of ODEs for the measure of lending relationships, credit market tightness, and the market-clearing condition for liquid assets are

$$\dot{\ell}_t = \alpha(\theta_t)(1-\ell_t)-\delta\ell_t \tag{89}$$

$$\dot{\theta}_t = \frac{\alpha(\theta_t)}{1-\epsilon(\theta_t)}\left\{(\rho+\delta)\frac{\theta_t}{\alpha(\theta_t)}+(1-\eta)\theta_t-\frac{\eta\Delta\pi(s_t)}{\zeta}\right\} \tag{90}$$

$$\mathrm{M}_t = \ell_t m^b\left(s_t\right)+(1-\ell_t)m^u\left(s_t\right), \tag{91}$$

where $m^u\left(s_t\right)$ and $m^b\left(s_t\right)$ are the implicit solutions to (15) and (19) with $s_t\equiv\rho-r_t$, and M_t is the supply of liquid assets. An equilibrium is a time path, $\{\ell_t,\theta_t,\mathrm{M}_t,s_t\}$, that solves (89)-(91) given ℓ_0 and monetary policy formulated either as s_t or M_t.

Appendix A4. Numerical procedures for optimal policy problems

Optimal policy with commitment

We solve for social welfare $\tilde{\mathbb{W}}(\ell,\theta)$ and the optimal policy function $\theta' = g_\theta(\ell,\theta)$ using value function iteration on a discrete grid.

1. Discretize the state space into $N_\ell \subseteq [0,1]$ and $N_\theta \subseteq \Omega$, where the lower and upper limits of Ω are given by solving for $\theta_t = \theta_{t+1} = \theta$ in 25 when s is either $s = 0$ or $s \to \infty$.
2. Define the Bellman operator as $T\tilde{\mathbb{W}}_n$ given by the right-hand side of (29).
3. Set the initial guess of $\tilde{\mathbb{W}}_0$ given by a policy consistent with a constant spread, s, defined as θ' in
$$\theta = \frac{\bar{\alpha}\beta\eta}{\zeta}\Delta\pi\left(s\right) + \beta(1-\delta) - 1 + \beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]\theta'.$$
4. Update $\tilde{\mathbb{W}}_{n+1}(\ell,\theta) = T\tilde{\mathbb{W}}_n(\ell,\theta)$
5. Repeat 4. until $\max_{\ell,\theta}\left\|\tilde{\mathbb{W}}_{n+1}(\ell,\theta) - \tilde{\mathbb{W}}_n(\ell,\theta)\right\| < \epsilon$. This sequence is Cauchy and converges to the unique fixed point of T.
6. Set $\mathbb{W}\left(\ell_0\right) = \max_{\theta\in\Omega}\tilde{\mathbb{W}}(\ell_0,\theta)$.

Optimal policy without commitment

We solve for the optimal policy functions $\Theta(\ell, m^{u'})$ and $\mathcal{K}(\ell)$ using contraction mappings on a discrete grid.

1. Discretize the state space into $N_\ell \subseteq [0,1]$ and $N_m \subseteq [0,k^*]$, where $y'^*) = 1$.
2. Set the initial guess of $\mathcal{K}_0(\ell)$ under a policy consistent with a constant spread, s, or $s = \lambda^u\left[y(\mathcal{K}(\ell)) - 1\right]$.
3. Given $\mathcal{K}_n(\ell)$, for any $n = 0,1,2,\ldots$, compute $\Theta_n(\ell,m^{u'})$ by iterating over the functional equation $T_{\theta,n}\Theta(\ell,m^{u'})$ given by the right-hand side of 31. Repeat until $\max_{\ell,m^{u'}}\left\|\Theta_{j+1}(\ell,m^{u'}) - \Theta_j(\ell,m^{u'})\right\| < \epsilon$. This sequence is Cauchy and converges to the unique fixed point of $T_{\theta,n}$, which is denoted as $\Theta_n(\ell,m^{u'})$.
4. Given $T_{\theta,n}$ from 3., update $\mathcal{K}_{n+1}(\ell)$ by solving the social welfare function $\mathcal{W}_n(\ell)$ using the functional equation $T_{\mathcal{W},n}\mathcal{W}(\ell)$ given by the right-hand side of 33. Repeat until $\max_\ell\left\|\mathcal{K}_{j+1}(\ell) - \mathcal{K}_j(\ell)\right\| < \epsilon$.
5. Check if $\left\|\mathcal{K}_n(\ell) - \mathcal{K}_{n+1}(\ell)\right\| < \epsilon$. If true, stop. Else, set $\mathcal{K}_n = \mathcal{K}_{n+1}$ and return to 3.

Optimal policy given temporary lockdown

Solve for the optimal policy function $g_\theta(\ell,\theta)$ and social welfare function $\tilde{\mathbb{W}}(\ell,\theta)$ under constant (λ^u,λ^b) as above. Given the length of the lockdown $T > 0$, solve for the optimal policy $g_{\theta,t}(\ell,\theta)$ and social welfare $\tilde{\mathbb{W}}_t(\ell,\theta)$ for $t = 0,\ldots,T$ using backward induction.

1. Let $\tilde{\mathbb{W}}_{T+1} = \tilde{\mathbb{W}}$.

2. For $t = 0, \ldots, T$, solve for $\tilde{\mathbb{W}}_t$ as

$$
\begin{aligned}
\widetilde{\mathbb{W}}_t(\ell, \theta) \quad = \quad & \max_{\theta' \in \Gamma_t(\theta)} \Big\{ -\zeta\theta(1-\ell) + \beta(1-\ell'^{u}\nu\left[y(m^{u'}) - m^{u'}\right] \\
& + \beta\ell'^{b} \left\{ \nu\left[y\left(k^{b'}\right) - k^{b'} - \psi(\mathrm{L}')\right] + (1-\nu)\left[y\left(\hat{k}\right) - \hat{k} - \psi(\hat{k})\right] \right\} \\
& + \beta\widetilde{\mathbb{W}}_{t+1}(\ell', \theta') \Big\},
\end{aligned}
\tag{92}
$$

where $\Gamma_t = [\underline{\theta}_t, \bar{\theta}_t]$ is defined iteratively by

$$
\begin{aligned}
\underline{\theta}_t = & \quad \tfrac{\bar{\alpha}\beta\eta}{\zeta}\Delta\pi(0) + \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]\underline{\theta}_{t+1} + \beta(1-\delta) - 1 \\
\bar{\theta}_t = & \quad \tfrac{\bar{\alpha}\beta\eta}{\zeta}\Delta\pi(+\infty) + \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]\bar{\theta}_{t+1} + \beta(1-\delta) - 1
\end{aligned}
$$

3. Set $\mathbb{W}_0(\ell_0) = \max_{\theta \in [\underline{\theta}_0, \bar{\theta}_0]} \widetilde{\mathbb{W}}_0(\ell_0, \theta)$.

Appendix A5: Ramsey problem under a timeless approach

Notice the Ramsey solution is not bound by any past promises at time $t = 0$, i.e., it is free to select any equilibrium, in our context by choosing any θ_0. To see how this matters, suppose the size of the credit shock approaches zero, i.e., the economy starts at its stationary solution. Letting the policymaker reset its policy is not innocuous, as illustrated in Figure **??**. The policymaker reduces s_1 even though there is no exogenous destruction of lending relationships, which paradoxically, generates a small reduction in ℓ.

Woodford (1999, 2003) proposed amending the Ramsey solution to discipline the initial choice of equilibrium. We adopt a similar notion. The solution to the Bellman equation (29) gives a policy function expressed as $\theta_{t+1} = \Theta^*(\ell_t, \theta_t)$. Suppose the economy has an infinite history. Substituting $\theta_t = \Theta^*(\ell_{t-1}, \theta_{t-1})$ and iterating, the policy choice at t can be expressed as a function of the entire history of lending relationships, i.e., $\theta_{t+1} = \Theta^\infty(\ell_t, \ell_{t-1}, \ell_{t-2}, ...)$ or, equivalently, $s_{t+1} = S^\infty(\ell_t, \ell_{t-1}, \ell_{t-2}, ...)$ where S^∞ is a time-invariant policy function that specifies the spread as a function of the entire history of lending relationships.

The Ramsey solution under a timeless approach sets θ_0 such that $\theta_0 = \Theta^\infty(\ell_{-1}, \ell_{-2}, \ell_{-3}, ...)$. If the economy was at a steady state, then $\ell_{-1} = \ell_{-2} = \ell_{-3} = \ell^*$ and $\theta_0 = \Theta^\infty(\ell^*, \ell^*, \ell^*, ...) = \theta^*$. Hence, $\theta_1 = \Theta^*(\ell_0, \theta^*)$ which determines s_1 from the free-entry condition. In particular, if the size of the shock is zero, $\ell_0 = \ell^*$, then $\theta_1 = \Theta^*(\ell^*, \theta^*) = \theta^*$ and the economy remains at its stationary solution.[32] Figure 11 plots the optimal timeless policy outcomes (dashed lines) for different sizes of the shock.

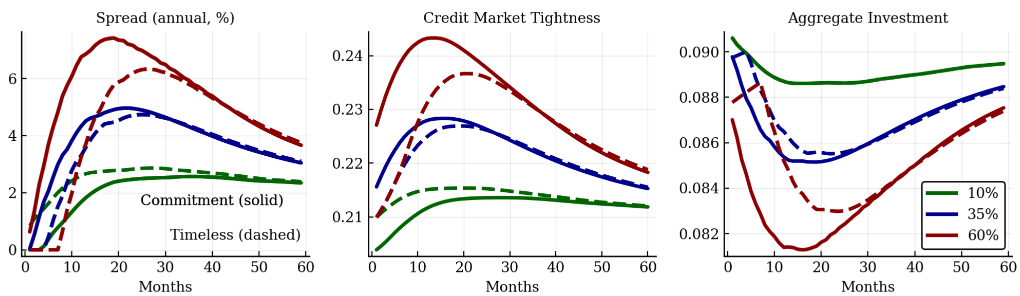

Figure 11: Optimal policy: unrestricted commitment (solid) vs. timeless approach (dashed)

The Ramsey solution under a timeless approach features a hump-shaped path for the spread. By restricting θ_0 to its steady-state value, the timeless approach impacts the initial spread set by the policymaker. For small shocks, s_0 is set close to its long-run value whereas for large shocks, s_0 is close to zero.

[32] As mentioned to us by Edouard Challe, in the case of a one-off, unexpected shock, the timeless perspective does not fit the description of a stationary, fully state-contingent policy plan where the policymaker has planned a long time ago how it would react to any future shock.

Appendix A6: Quantitative robustness

In the following, we discuss how optimal monetary policy depends on some key parameters of the model. In the main calibration, we find ν very close to one. In the first robustness check, we lower ν to imply twice the annual probability of theft, $1-\nu$, from 2.5% to 5%. Another key parameter of the model is banks' bargaining power that determines whether there is too much or too little creation of relationships. In our benchmark calibration, $\eta = 0.52$. Our second robustness check reports optimal monetary policy when $\eta \in \{0.25, 0.75\}$. Our third check shuts down the cost of monitoring loans, $B = 0$. In that case, the only cost of external finance is the cost of bank entry. In each case, we keep the other parameters of the model unchanged and report them in the table below.

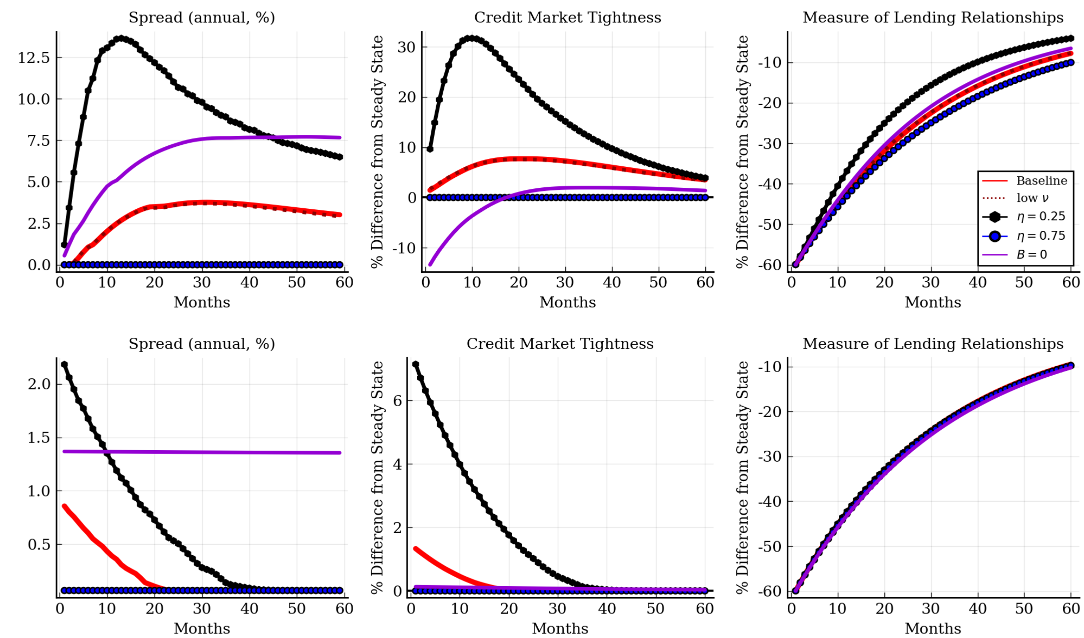

Figure 12: Robustness of optimal policy with commitment (top) and without commitment (bottom)

Figure 12 plots the optimal policy outcomes following a 60% destruction of relationships under commitment (top panels) and without commitment (bottom panels). The main takeaway is that the qualitative features of our baseline example are robust. The time path of the interest spread is hump-shaped under the Ramsey solution while it is downward-sloping in the absence of commitment. Further, the magnitude of the spreads tend to be higher with commitment, and the recovery of lending relationships is faster. There are two important caveats. When bank bargaining power is high enough, optimal policy should set the spread to the lower bound. Secondly, when there is no cost to external finance, the planner maintains a spread above zero under both commitment and without commitment.

Specification	ρ	ξ	δ	ν	$\bar{\alpha}$	λ^u	λ^b	B	η	ζ
Baseline	0.0033	8.6	0.01	0.998	0.09	0.126	0.144	6.19	0.52	0.024
$\nu = 0.995$	0.0033	8.6	0.01	0.995	0.09	0.126	0.144	6.19	0.52	0.024
$\eta = 0.250$	0.0033	8.6	0.01	0.998	0.09	0.126	0.144	6.19	0.25	0.024
$\eta = 0.750$	0.0033	8.6	0.01	0.998	0.09	0.126	0.144	6.19	0.75	0.024
$B = 0$	0.0033	8.6	0.01	0.998	0.09	0.126	0.144	0.00	0.52	0.024